Neural Networks and Learning Theory

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Exercise Sheet 2
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Exercise 1.

Let C_1 and C_2 be two finite pointsets of $\mathbb{R}^n$ such that there are some $b \in \mathbb{R}$ and some $z \in \mathbb{R}^n$ with:

$$z^T \cdot x \geq b \;\; \forall\, x \in C_1 \;\; , \;\; z^T \cdot x < b \;\;\; \forall\, x \in C_2.$$

(a) Show that you can assume without loss of generality that b is equal to 1.

(b) Prove that there exist a vector $\tilde{z} \in \mathbb{R}^n$ and a real positive number $c > 0$ such that:

$$\tilde{z}^T \cdot x \geq 1 + c \;\; \forall\, x \in C_1 \;\; , \;\;\; \tilde{z}^T \cdot x \leq 1 - c \;\; \forall\, x \in C_2.$$

Exercise 2.

Let C_1, C_2 like in **Exercise 1**. Furthermore, there are $\tilde{z} \in \mathbb{R}^n$ and $c > 0$ with:

$$\tilde{z}^T \cdot x \geq 1 + c \;\; \forall\, x \in C_1 \;\; , \;\;\; \tilde{z}^T \cdot x \leq 1 - c \;\; \forall\, x \in C_2.$$

Show that:

$$inf\{\|x - y\|_2 \mid x \in C_1, y \in C_2\} \;\geq\; \frac{2c}{\|\tilde{z}\|_2},$$

where $\|\tilde{z}\|_2 := \sqrt{\sum_{i=1}^{n} \tilde{z}_i^2}$ denotes the **Euclidean norm** of $\tilde{z} \in \mathbb{R}^n$.

Exercise 3.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function of parameter vector x. The **Derivative** of $f(x)$ with respect to x is defined by the vector (we disregard here a more mathematical approach for defining the Derivative):

$$Df(x) = \frac{\partial f(x)}{\partial x} := \left(\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n} \right).$$

Using above definiton, prove that the following equalities hold:

(a)

$$\frac{\partial h^T \cdot x}{\partial x} = \frac{\partial x^T \cdot h}{\partial x} = h$$

for all vectors $x, h \in \mathbb{R}^n$.

(b)

$$\frac{\partial x^T \cdot M \cdot x}{\partial x} = 2 \cdot M \cdot x$$

for some symmetric matrix $M \in \mathbb{R}^{n \times n}$, i.e. $M^T = M$.

(c)
$$\frac{\partial \|x\|_2}{\partial x} = \frac{x}{\|x\|_2},$$
for all $x \in \mathbb{R}^n \setminus \{0\}$.

Exercise 4.

Using the method of **Lagrange multipliers**, find the points (x, y) on the ellipse $x^2 + xy + y^2 - 3 = 0$ with minimum and maximum distance to the origin $(0, 0)$.

(a) Formulate this problem as an optimization problem, i.e. find the objective function f to be minimized (maximized) and the conditions (a function $g = 0$) of the optimization problem.

(b) Define the **Lagrange function** $L(x, y, \lambda) := f(x, y) - \lambda g(x, y)$
and solve the equation $DL(x, y, \lambda) = 0$.

(c) Replace the Solution of the obove equation in the condition $g(x, y) = 0$ to find the the maxima and minima.

Exercise 5.

(a) Show that $f(x) := \frac{1}{2}x^T x$ is a convex function.

(b) Show that the affine function $f(x) := b^T x + c, b \in \mathbb{R}^n, c \in \mathbb{R}$ is convex.

(c) Show that: for a symmetric matrix $M \in \mathbb{R}^{n \times n}$ is the function
$$f(x) := x^T M x$$
convex if and only if the matrix M is positive semi-definite, i.e. $\forall\ h \in \mathbb{R}^n$ is $h^T \cdot M \cdot h \geq 0$.