Algebraische Rechenmodelle, exercise sheet 10

January 26, 2015

Exercise 1

Let M be a BSS machine over $\mathbb{R}$ and let $\alpha := (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$ be the machine constants of M. Let $\phi_n^+(M)(\alpha, x_1, \ldots, x_n)$ be a first order formula which is true iff M accepts $x \in \mathbb{R}^n$ and let $\phi_n^-(M)(\alpha, x)$ be a first order formula which is true iff M rejects $x \in \mathbb{R}^n$. Let $\tilde{\mathbb{R}}$ be a real closed extension of $\mathbb{R}$ and define

$$\begin{aligned} x \in \widetilde{A_n} &\Leftrightarrow \phi_n^+(M)(\alpha, x_1, \ldots, x_n) \\ x \in comp(\widetilde{A_n}) &\Leftrightarrow \phi_n^-(\alpha, x_1, \ldots, x_n). \end{aligned}$$

Use the theorem below to show that $\widetilde{A_n}$ and $comp(\widetilde{A_n})$ are disjoint and that $\widetilde{A_n} \cup comp(\widetilde{A_n}) = \tilde{R}$. Show then that their definition does not depend on the machine M used to decide A over $\mathbb{R}$, i.e, any other machine deciding A over the reals will result in the same subsets of $(\tilde{R})^n$.

Theorem 1 (transfer principle)**.** *Let $(R, \leq)$ and $\tilde{R}, \tilde{\leq})$ be real closed fields such that $R \subset \tilde{R}$ and $\tilde{\leq}$ extends the ordering $\leq$ (i.e. $\forall x, y \in R$ we have $x \leq y$ iff $x \tilde{\leq} y$). Let ψ be a first order sentence (i.e. no free variables) which has only constants from R. Then ψ holds in $\tilde{R}, \tilde{\leq})$ if and only if ψ holds in $(R, \leq)$*

Exercise 2

Let F be a subfield of the reals: $\mathbb{Q} \subseteq F \subseteq \mathbb{R}$. Call a set $A \subseteq \mathbb{R}$ *countably semi-algebraic over F* iff it is a countable union $A = \bigcup_{m \in \mathbb{N}} A_m$ of semi-algebraic sets A_m that are definable over F. Show that a language $L \subseteq \mathbb{R}^\infty$ is semi-decidable in the BSS model over $\mathbb{R}$ if and only if L is countably semi-algebraic over some *finitely generated* field extension of $\mathbb{Q}$.

Exercise 3

Let $\tilde{\mathbb{R}}$ be a real closed extension of $\mathbb{R}$. Prove that $P_{\tilde{\mathbb{R}}} \neq NP_{\tilde{\mathbb{R}}} \Rightarrow P_{\mathbb{R}} \neq NP_{\mathbb{R}}$.

Hint: Use the fact that the problem (F^4, F^4_{zero}) over $\mathbb{R}$ is $NP_{\mathbb{R}}$-complete and the problem (F^4, F^4_{zero}) over $\tilde{\mathbb{R}}$ is $NP_{\tilde{\mathbb{R}}}$-complete. Show with help of the transfer principle that a machine M deciding the problem (F^4, F^4_{zero}) over $\mathbb{R}$ can be reinterpreted as a machine over $\tilde{R}$ and then it decides the problem (F^4, F^4_{zero}) over $\tilde{\mathbb{R}}$ within the same time bound.