Algebraische Rechenmodelle, exercise sheet 8

January 8, 2015

Exercise 1

Prove that every ideal in $\mathbb{C}[x]$, $x = (x_1, \dots, x_n)$ is finitely generated. Hint: use induction on n, assume that some ideal $I \in \mathbb{C}[x_1, \dots, x_{n+1}]$ is not finitely generated and derive a contradiction. Show that there exists a sequence $f_0, f_1, \dots \in I$ with the property that for all m, f_m is a function in $I \setminus \mathcal{I}(f_0, \dots, f_{m-1})$ of minimal degree in x_{n+1}. Let $a_i \in \mathbb{C}[x_1, \dots, x_n]$ be the leading coefficient of f_i considered as polynomial in x_{n+1} with coefficients in $\mathbb{C}[x_1, \dots, x_n]$. Now use the induction hypothesis to construct for some $M \in \mathbb{N}$ a function $g \in \mathcal{I}(f_0, \dots, f_{M-1})$ with the same leading coefficient as f_M and show that this contradicts the properties of the sequence $f_0, f_1, \dots \in I$. Alternatively, you can also search the internet for "Hilbert's basis theorem".

Exercise 2

Let $\leq$ be an arbitrary monomial ordering on $\mathbb{N}_0^n$.

a) Does there exist a smallest element, i.e., is there an $\alpha \in \mathbb{N}_0^n$ such that for all $\beta \in \mathbb{N}_0^n$, $\alpha \leq \beta$?.

b) Does $x^\alpha | x^\beta$ imply that $\alpha \leq \beta$?

Exercise 3

Show that division of $f(x, y) = x^3y^2 - 2xy^4$ by $f_1(x, y) := x^2y^2 - x$ and $f_2(x, y) := xy^2 + y$ with the lexicographic order $(x < y)$ does not have a uniquely determined rest.