Algebraische Rechenmodelle, exercise sheet 7

January 5, 2015

We want the prove the following theorem.

Theorem 1. *Let $f \in k[X_1, \ldots, X_n]$ have expression size u. Then f is a projection of PER_{2u+2}.*

We first prove a useful lemma.

Lemma 1. *Let R be a commutative ring. For $i = 1, 2$ let $A_i \in R^{d_i \times d_i}$ be an upper triangular matrix with 1's on the diagonal, $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \ldots) \in R^{1 \times d_i}$ and $\beta_i = (\beta_{i1}, \beta_{i2}, \ldots)^\top \in R^{d_i \times 1}$. Then for the matrices*

$$M_1 := \begin{pmatrix} \alpha_1 & 0 \\ A_1 & \beta_1 \end{pmatrix}, M_2 := \begin{pmatrix} \alpha_2 & 0 \\ A_2 & \beta_2 \end{pmatrix}, M := \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ A_1 & 0 & \beta_1 \\ 0 & A_2 & \beta_2 \end{pmatrix},$$

we have

$$per(M) = per(M_1) + per(M_2).$$

Let $M[i|j]$ denote the matrix obtained by deleting in M row i and column j. Here are some hints for finding a proof.

- use induction on d_2;
- show that the equations below hold.

$$\begin{aligned} \mathrm{per}(M_2) &= \alpha_{21}\mathrm{per}(M_2[1|1]) + \mathrm{per}(M_2[2|1]) && (1) \\ \mathrm{per}(M[1|d_1+1]) &= \mathrm{per}(M_2[1|1]) && (2) \\ \mathrm{per}(M[d_1+2|d_1+1]) &= \mathrm{per}(M_1) + \mathrm{per}(M_2[2|1]) && (3) \\ \mathrm{per}(M) &= \alpha_{21}\mathrm{per}(M[1|d_1+1]) + \mathrm{per}(M[d_1+2|d_1+1]) && (4) \end{aligned}$$

Let us now prove the theorem. Let $\mathcal{E}$ be the set of expressions over $I = k \cup \{X_1, \ldots, X_n\}$. Recursively define a mapping $\mu : \mathcal{E} \to \cup_{s \geq 1} I^{s \times s}$ with the following properties, for all $\phi \in \mathcal{E}$:

(A) $\mathrm{val}(\phi) = \mathrm{per}(\mu(\phi))$.

(B) If ϕ has expression size u, then $\mu(\phi)$ has size $s \times s$ with $s = 2u + 2$.

(C) There exist $A \in I^{(s-1)\times(s-1)}$, $\alpha \in I^{1\times(s-1)}$, $\beta \in I^{(s-1)\times 1}$, with $s = 2u + 2$ as in (B), such that A is upper triangular with 1's on the diagonal and

$$\mu(\phi) = \begin{pmatrix} \alpha & 0 \\ A & \beta \end{pmatrix}$$

(D) $\mu(\phi)$ has in each column at most one entry which is an indeterminate. The last column contains no indeterminate