INVESTIGATING RAYLEIGH–BÉNARD CONVECTION AT LOW PRANDTL NUMBERS USING ONE-DIMENSIONAL TURBULENCE MODELING

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ABSTRACT

We numerically investigate the heat transfer in turbulent Rayleigh–Bénard convection at two Prandtl numbers, $Pr = 0.021$ and 0.7, respectively. Small-scale resolving simulations up to the Rayleigh numbers $Ra = 10^{13}$ ($Pr = 0.021$) and 10^{16} ($Pr = 0.7$) are made feasible by utilizing the stochastic, one-dimensional turbulence (ODT) model. Present ODT simulations exhibit effective Nusselt number Nu scalings of the form $Nu \sim Ra^{\gamma}$. At low Rayleigh numbers, ODT yields a scaling exponent of $\gamma = 0.29$ ($Pr = 0.021$) and 0.32 ($Pr = 0.7$), respectively. Both values are systematically, but just slightly, overestimating available reference data. At high Rayleigh numbers, present ODT results exhibit an increase of the exponent to $\gamma = 0.32$ ($Pr = 0.021$) and 0.36 ($Pr = 0.7$), respectively. Our results suggest that ODT is able to capture a transition from the classical to the ultimate state of convection in terms of (i) critical Rayleigh number and (ii) increase of γ.

INTRODUCTION

We study turbulent Rayleigh–Bénard (RB) convection as a canonical problem for buoyancy-driven flows, which is relevant for various technological and geophysical applications (e.g. Chillà & Schumacher, 2012). The setup is sketched in Figure 1 and given by a cylindrical cell with gap width L, radius R, and aspect ratio $\Gamma = 2R/L$ (with $\Gamma \to \infty$ in ODT). Fluid is confined between the heated wall (hw) at the bottom and the cooled wall (cw) at the top.

RB convection is governed by two dimensionless control parameters, the Rayleigh number Ra and the Prandtl number Pr; the Nusselt number Nu is a simulation result:

$$Ra = \frac{g\,\beta\,\Delta T\,L^3}{\nu\kappa} \qquad Pr = \frac{\nu}{\kappa} \qquad Nu = \frac{Q}{Q_\mathrm{c}} \qquad (1)$$

Here, g is the gravity, $\Delta T = T_\mathrm{hw} - T_\mathrm{cw}$ is the imposed temperature difference, β is the thermal expansion coefficient, ν the kinematic viscosity, and κ the thermal diffusion coefficient of the working fluid. Nu is defined as the ratio of the total heat transfer Q and the conductive heat transfer Q_c without convection. In applications, one has $Pr \simeq 1$ for many gases and $Pr \lesssim 10^{-2}$ for liquid metals, whereas Ra can easily reach up to 10^{27} (Chillà & Schumacher, 2012). We thus focus on the Ra-dependency for fixed Pr.

For sufficiently large Ra, a transition from the classical to the ultimate state of convection should manifest itself by a change of the similarity scaling $Nu \sim Ra^{\gamma}$ (Kraichnan, 1962). There is evidence from laboratory measurements (Chillà & Schumacher, 2012; He *et al.*, 2012) and two-dimensional direct numerical simulations (2-D DNSs; Zhu *et al.*, 2018) for this transition to occur at $Ra \sim 10^{14}$ for $Pr \simeq 1$. 3-D DNSs have remained in the classical state by reaching $Ra = 2 \times 10^{12}$ for $Pr = 0.7$ (Stevens *et al.*, 2011). The transition is expected to occur at lower Ra for smaller Pr, that is, $Ra \sim 10^{11}$ for $Pr = 0.021$. However, 3-D DNSs could not reach the transition either ($Ra = 4 \times 10^8$ for $Pr = 0.021$; Schumacher *et al.*, 2016).

We believe that stochastic modeling strategies might help to alleviate the limited Ra-range in 3-D simulations without introducing an inverse energy cascade as in the 2-D simulations. This is addressed here by utilizing the one-dimensional turbulence (ODT) model extending a previous study by Wunsch & Kerstein (2005). Here we make use of a numerically efficient, adaptive implementation of ODT (Lignell *et al.*, 2013).

ONE-DIMENSIONAL TURBULENCE

ODT is a stochastic turbulence model that resolves all scales of a turbulent flow but with reduced dimensionality (Kerstein, 1999). The computational domain is a single line, which points vertically in the direction of the largest mean gradients (Figure 1).

Here, we limit our attention to weak density variations and take a linear equation of state (Oberbeck–Boussinesq approximation), $\rho(T) = \rho_0 \left[1 - \beta\,(T - T_0)\right]$, where ρ is the density and T is the temperature; the subscript 0 denotes reference values. The ODT governing equations take the form (Kerstein *et al.*, 2001; Wunsch & Kerstein, 2005):

$$\frac{\partial u_i}{\partial t} + \mathscr{E}_{u,i} = \nu \frac{\partial^2 u_i}{\partial z^2} \qquad \frac{\partial T}{\partial t} + \mathscr{E}_T = \kappa \frac{\partial^2 T}{\partial z^2} \tag{2}$$

Here, u_i is the ith velocity component, t is the time and z is the vertical coordinate. $\mathscr{E}_{u,i}(u_j,T)$ and $\mathscr{E}_T(u_j,T)$ are stochastic terms modeling the nonlinear turbulent advection. Note that $\mathscr{E}_{u,i}$ contains momentum sources due to buoyancy (Wunsch & Kerstein, 2005) and fluctuating pressure-velocity couplings (Kerstein *et al.*, 2001).

The temperature is prescribed by isothermal wall boundary conditions and no-slip conditions are used for the velocity. There are no other domain boundaries in ODT.

The deterministic evolution (diffusion) is interrupted by stochastic mapping events, so-called eddy events, which model turbulent advection in one resolved dimension with a measure-preserving map (Kerstein, 1999). A thinning-and-rejection method is used to find plausible eddy events (Kerstein, 1999). The eddy timescale τ plays a central role in the evaluation of physical soundness and is given by Wunsch & Kerstein (2005); Lignell *et al.* (2013) as:

$$\frac{1}{\tau} = C\sqrt{\frac{2}{\rho_0 l^5}\left(\Delta E_{\mathrm{kin}} + \Delta E_{\mathrm{pot}} - Z E_{\mathrm{vp}}\right)} \tag{3}$$

Here, ΔE_{kin} and ΔE_{pot} denote the changes of the kinetic and the potential energy due to the implementation of an eddy of size (scale) l. E_{vp} is a viscous penalty energy related to the Kolmogorov scale. The coefficients C and Z are model parameters which need to be determined with reference data. We matched Nu to the 3-D DNS data of Stevens *et al.* (2011) and Scheel & Schumacher (2014, 2016) for a single Ra at each Pr. This resulted in $C = 43$ ($Pr = 0.021$), $C = 60$ ($Pr = 0.7$), and $Z = 220$ (both Pr).

RESULTS

Figure 2 shows profiles of the instantaneous and mean temperature together with profiles of a wall-parallel velocity component for the two Prandtl numbers investigated. For $Pr \ll 1$, the thermal diffusion is much faster than the momentum diffusion so that the temperature exhibits larger spatial scales in the case of $Pr = 0.021$ compared to $Pr = 0.7$. The spatial scales in the ODT velocity fields are comparable for both Pr and similar to those in the temperature field at $Pr = 0.7$. This is due to similar Grashof numbers $Gr = Ra/Pr = (1.6 \pm 0.3) \times 10^{10}$.

Figure 3 shows the Nusselt number over the Rayleigh number for both Prandtl numbers investigated. Each filled symbol corresponds to a temporal ODT simulation running for $O(10^6)$ eddy implementations in the statistically stationary state. The Nusselt number is computed via the mean wall temperature gradient, $Nu = |\mathrm{d}\langle T\rangle/\mathrm{d}z|_{\mathrm{w}}/(\Delta T/L)$. Error bars due to the evaluation at the heated or cooled wall are within the symbol size. Scaling laws of the form $Nu \sim Ra^\gamma$ describe the ODT results for low and high Rayleigh numbers. The values of γ obtained with a least-squares fit are given in Table 1. The transitional Ra range agrees with reference data from He *et al.* (2012, $Pr = 0.7$) and Scheel & Schumacher (2016, $Pr = 0.021$).

Table 1. Similarity scaling $Nu \sim Ra^\gamma$. Reference values are from Scheel & Schumacher (2014, 2016).

Pr	Ra	γ_{ODT}	γ_{ref}
0.021	$\leq 10^{10}$	0.29 ± 0.01	0.26 ± 0.01
0.021	$\geq 10^{11}$	0.32 ± 0.01	—
0.7	$\leq 10^{13}$	0.32 ± 0.01	0.29 ± 0.01
0.7	$\geq 4 \times 10^{14}$	0.36 ± 0.01	—

CONCLUSION

Stochastic ODT simulations of low-Pr convection have been conducted. The Nusselt number is found to scale like $Nu \sim Ra^\gamma$, where γ increases between $10^9 < Ra < 10^{11}$ ($Pr = 0.021$) and $10^{13} < Ra < 10^{15}$ ($Pr = 0.7$). This hints at a transition from the classical to the ultimate state. ODT simulations at other Pr are under way to analyze this further.

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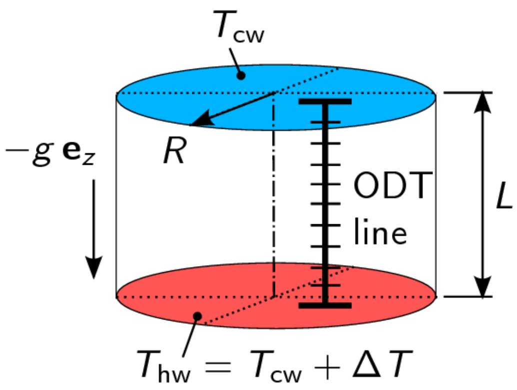

Figure 1. Schematic of the RB setup. The ODT computational domain (ODT line) is a representative vertical line.

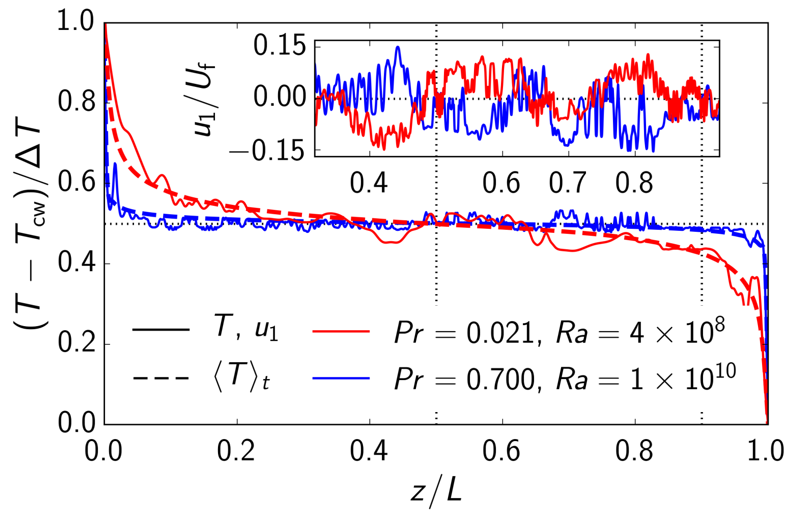

Figure 2. Representative vertical profiles simulated with ODT. Profiles of the mean temperature $\langle T \rangle$, the instantaneous temperature T, and one representative wall-parallel velocity component u_1 (inset) are shown for two cases with similar Grashof number ($Gr = Ra/Pr = (1.6 \pm 0.3) \times 10^{10}$). The gap width L, the temperature difference ΔT, and the free-fall velocity $U_f = \sqrt{g\,\beta\,\Delta T\,L}$ are used as reference scales. Dotted lines are given for orientation.

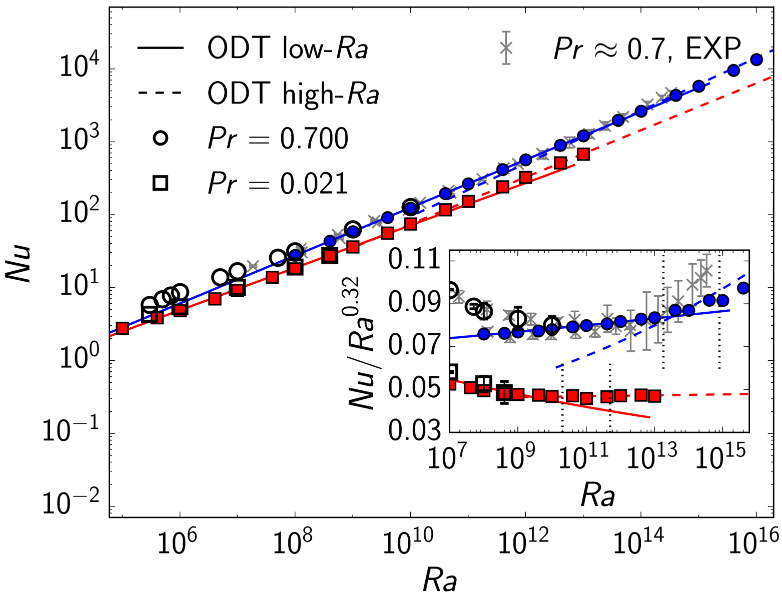

Figure 3. Scaling of the Nusselt number Nu versus Rayleigh number Ra for two Prandtl numbers Pr. Present ODT results (colored symbols and lines) are compared to DNS results (empty symbols; Scheel & Schumacher, 2014, 2016). Experimental data (EXP, gray symbols) are from various sources compiled in Chillà & Schumacher (2012). Invisible error bars are within the symbol size. The inset shows the same data but compensated with $Ra^{0.32}$. Dotted lines indicate the anticipated transition range from the classical to the ultimate state (He *et al.*, 2012; Scheel & Schumacher, 2016).