Generalized Parameter Estimation-based Observers: Application to Power Systems and Chemical-Biological Reactors

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Abstract

In this paper we propose a new state observer design technique for nonlinear systems. It consists of an extension of the recently introduced parameter estimation-based observer, which is applicable for systems verifying a particular algebraic constraint. In contrast to the previous observer, the new one avoids the need of implementing an open loop integration that may stymie its practical application. We give two versions of this observer, one that ensures asymptotic convergence and the second one that achieves convergence in finite time. In both cases, the required excitation conditions are strictly weaker than the classical persistent of excitation assumption. It is shown that the proposed technique is applicable to the practically important examples of multimachine power systems and chemical-biological reactors.

Key words: Estimation parameters, nonlinear systems, observers, time-invariant systems, power systems.

1 Problem Formulation

In this paper we are interested in the design of state observers for nonlinear control systems whose dynamics is described by

$$\dot{x} = f(x,u),\ y = h(x,u), \tag{1}$$

where $x \in \mathbb{R}^n$ is the systems state, $u \in \mathbb{R}^m$ is the control signal and $y \in \mathbb{R}^p$ are the *measurable* output signals. Similarly to all mappings in the paper, the mappings $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,\ h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ are assumed smooth. The problem is to design a dynamical system

$$\dot{\chi} = F(\chi, y, u),\ \hat{x} = H(\chi, y, u) \tag{2}$$

with $\chi \in \mathbb{R}^{n_\chi}$, such that *for all* initial conditions $x(0) \in \mathbb{R}^n,\ \chi(0) \in \mathbb{R}^{n_\chi}$,

$$\lim_{t\to\infty} |\hat{x}(t) - x(t)| = 0, \tag{3}$$

where $|\cdot|$ is the Euclidean norm. We are also interested in the case when the observer ensures *finite convergence time* (FCT), that is, when there exists $t_c \in [0, \infty)$ such that

$$\hat{x}(t) = x(t),\ \forall t \geq t_c. \tag{4}$$

Following standard practice in observer theory [5] we assume that u is such that the state trajectories of (1) are bounded. Since the publication of the seminal paper [16], which dealt with linear time-invariant (LTI) systems, this problem has been extensively studied in the control literature. We refer the reader to [3,5,7,10] for a review of the literature. In this paper we propose an

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extension of the parameter estimation-based observer (PEBO) design technique reported in [21]. The main novelty of PEBO is that it translates the task of state observation into an on-line *parameter estimation* problem.

The main features of the new observer design technique proposed in the paper, called generalized PEBO (GPEBO), are the following.

(F1) The key "transformability into cascade form" condition of the original PEBO [21, Assumption 1] is *relaxed*, replacing it by a "transformability into state-affine form" discussed in [5, Chapter 3].

(F2) We identify a class of systems for which the second key condition of PEBO [21, Assumption 2]—which relates with the, far from obvious, solution of the parameter estimation problem—is obviated. The class is identified via a particular *algebraic constraint*.

(F3) It avoids the need of *open-loop integration* which stymies the practical application of this observer for systems subject to high noise environments—see [21, Remark R5].

(F4) Via the utilization of the fundamental matrix of an associated linear time-varying (LTV) system, the *signal excitation* needed to estimate the parameters is improved.

(F5) Using the dynamic regressor extension and mixing (DREM) procedure [2], which is a novel, powerful, parameter estimation technique, we propose a variation of GPEBO achieving FCT, that is, for which (4) holds, under the weakest sufficient excitation assumption [12]. [1]

(F6) It is proven that the conditions (F1) and (F2) are satisfied by the practically important case of multimachine power systems, while (F1) is verified by chemical-biological reactors.

For the multimachine power systems we consider the well-know three-dimensional "flux-decay" model of a large-scale power system [14,29], consisting of N generators interconnected through a transmission network, which we assume to be lossy, that is, we explicitly take into account the presence of transfer conductances. We prove that, using the *measurements* of active and reactive power—which is a reasonable assumption given the current technology [13,29]—as well as the rotor angle at each generator, the application of GPEBO allows us to recover *the full state* of the system, even in the presence of lossy lines. To the best of the authors' knowledge, this is the first globally convergent solution to the problem.

For the reaction problem we consider the classical dynamical model of the concentration components, *e.g.*, equation (1.43) in [4, Section 1.5], which describes the behavior of a large class of chemical and bio-chemical reaction systems. We propose a state observer that, in contrast with the standard asymptotic observers [4,8], has a *tunable* convergence rate. Similarly to the case of power systems, using DREM, we can ensure FCT for the particular case when the reaction rates are linear in the unmeasurable states.

The remainder of the paper is organized as follows. In Section 2, to place in context the contributions of GPEBO, we briefly recall the basic principles of PEBO. In Section 3 we give the main results. Section 4 is devoted to some discussion. Section 5 presents the application of the observer to two practical problems. The paper is wrapped-up with concluding remarks in Section 6. The proofs of the main propositions are given in appendices at the end of the paper.

Caveat This is an abridged version of the full paper [20].

2 Review of PEBO and Introduction to GPEBO

To make the paper self-contained, in this section we briefly recall the underlying principle of our previous PEBO design [21]. Then, with PEBO as the background, we highlight the main results of GPEBO, that extend its domain of applicability.

2.1 Basic construction of PEBO

As explained in the Introduction, the specificity of PEBO is that the problem of state observation is translated into a problem of *parameter estimation*, namely the initial conditions of the system (1). To achieve this objective, we consider in PEBO nonlinear systems of the form (1) that can be transformed, via a change of coordinates, to a *cascade* form. Let us assume for simplicity that the system is already given in this form, namely that $\dot{x} = B(u,y)$, where $B : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$. In PEBO we do an open-loop integration of $B(u,y)$, that is we define

$$\dot{\xi} = B(u,y), \tag{5}$$

an operation that has well-known shortcomings—see [21, Remark R5]—and make the observation that $\dot{x} = \dot{\xi}$, hence $x(t) = \xi(t) + \theta$ with $\theta := x(0) = \xi(0)$. Then, construct the observed state as $\hat{x} = \xi + \hat{\theta}$, with $\hat{\theta}$ and estimate of the unknown vector θ using the information of y. Except for the case when θ enters linearly the task of generating a consistent estimate for θ is far from trivial.

2.2 New construction of GPEBO

A first important difference of GPEBO is that we relax the assumption of transformability to a cascade form to transformability to an *affine-in-the-state* form

$$\dot{x} = \Lambda(u,y)x + B(u,y)$$

where $\Lambda : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^{n\times n}$. See [5, Chapter 3] for a discussion on these normal forms and the existing observer designs for them.

In GPEBO the fragile step of open-loop integration of

[1] See [23] for an FCT version of DREM, [24] for an interpretation as a Luenberger observer and [19,25] for two recent applications of DREM+PEBO techniques.

PEBO is replaced by the construction of a "copy" of the system via another dynamical system

$$\dot{\xi} = \Lambda(u,y)\xi + B(u,y).$$

avoiding the open-loop integration. The new idea introduced in GPEBO is to exploit the properties of the *fundamental matrix* of an LTV system as follows. Let us define the signal

$$e = x - \xi, \tag{6}$$

whose dynamics is described by the LTV system

$$\dot{e} = A(t)e \tag{7}$$

where $A(t) := \Lambda(u(t), y(t))$. As shown in all textbooks of linear systems theory a property of LTV systems is that all solutions of (7) can be expressed as linear combinations of the columns of its fundamental matrix, which is the unique solution of the matrix equation

$$\dot{\Phi}_A = A(t)\Phi_A, \;\Phi_A(0) = \Phi_A^0 \in \mathbb{R}^{n\times n},$$

with Φ_A^0 full-rank, see [27, Property 4.4]. More precisely,

$$e(t) = \Phi_A(t)[\Phi_A^0]^{-1}e(0).$$

Similarly to PEBO, in GPEBO we treat $e(0)$ as an unknown parameter $\theta := e(0)$, that we try to *estimate*. Invoking (6), the observed state is then generated as

$$\hat{x} = \xi + \Phi_A\hat{\theta}, \tag{8}$$

where, to simplify notation and without loss of generality, we set $\Phi_A^0 = I_n$, with I_n the $n \times n$ identity matrix. The use of the fundamental matrix is the *key step* of GPEBO.

Another important advantage of GPEBO is that, if the output mapping $h(x,u)$ of (1), can also be expressed in an affine in the state form,[2] that is,

$$h(x,u) = \mathtt{C}(u,y)x + \mathtt{D}(u,y), \tag{9}$$

then it is possible to obtain a *linear regressor equation* (LRE) for the unknown vector θ. Indeed, from the derivations above we get the LRE $\mathtt{y} = \psi\theta$ where we defined

$$\mathtt{y} := y - \mathtt{D}(u,y) - \mathtt{C}(u,y)\xi,\; \psi := \mathtt{C}(u,y)\Phi_A.$$

This is also a fundamental feature since, as it is well-known [15,28], the design of parameter estimators for LRE is a well-understood problem.

A final advantage of GPEBO over PEBO pertains to the excitation conditions needed for parameter estimation. Notice that, if in PEBO the mapping $h(x,u)$ satisfies the assumption (9) we can also obtain a LRE $\mathtt{y} = \psi\theta$, but with $\psi = \mathtt{C}(u,y)$. It is well-known that the convergence of all estimators is determined by the excitation of the regressor ψ. The presence of the additional term Φ_A in the regressor of GPEBO improves the excitation level. To appreciate this, consider the case when $\mathtt{C}(u,y)$ is *constant*. In that case it is impossible to estimate the parameter θ with the LRE of PEBO.

[2] In our main result we consider a more general assumption, but here we use this simple one for the sake of clarity.

3 Main Results

The GPEBO designs are based on the following two propositions. For ease of presentation we consider the case where we are interested in observing *all* state variables. In many applications it is only necessary to reconstruct *some* of these state variables, a case that can be treated with slight modifications to these propositions. Also, we present first the version of GPEBO that ensures *asymptotic* convergence and then, in Proposition 3, the one ensuring FCT. The proofs of both propositions are given in Appendices A and B, respectively.

3.1 An asymptotically convergent GPEBO

Proposition 1 Consider the system (1). Assume there exist mappings

$$\phi : \mathbb{R}^n \to \mathbb{R}^n,\; \phi^{\mathsf{L}} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n,\; B : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n,$$
$$\Lambda : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^{n\times n},\; L : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^{n\times n},$$
$$C : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$$

satisfying the following:

(i) The GPEBO partial differential equation (PDE)

$$\nabla\phi^\top(x)f(x,u) = \Lambda(u,h(x,u))\phi(x) + B(u,h(x,u)), \tag{10}$$

where $\nabla := (\frac{\partial}{\partial x})^\top$.

(ii) ϕ^{L} is a "left inverse" of ϕ, in the sense that it satisfies

$$\phi^{\mathsf{L}}(\phi(x), h(x,u)) = x. \tag{11}$$

(iii) The algebraic constraint

$$L(u,h(x,u))\phi(x) = C(u,h(x,u)), \tag{12}$$

is satisfied.

(iv) For the given u, all solutions of the LTV system

$$\dot{z} = \Lambda(u(t),y(t))z,$$

with y generated by (1), are *bounded*.

The GPEBO dynamics

$$\dot{\xi} = \Lambda(u,y)\xi + B(u,y) \quad (13a)$$
$$\dot{\Phi}_\Lambda = \Lambda(u,y)\Phi_\Lambda,\ \Phi_\Lambda(0) = I_n \quad (13b)$$
$$\dot{Y} = -\lambda Y + \lambda \Psi^\top [C(u,y) - L(u,y)\xi] \quad (13c)$$
$$\dot{\Omega} = -\lambda\Omega + \lambda\Phi_\Lambda\Phi_\Lambda^\top \quad (13d)$$
$$\dot{\hat{\theta}} = -\gamma\Delta(\Delta\hat{\theta} - \mathcal{Y}), \quad (13e)$$

with $\lambda > 0$ and $\gamma > 0$, with the definitions

$$\Psi := L(u,y)\Phi_\Lambda \quad (14a)$$
$$\mathcal{Y} := \text{adj}\{\Omega\}Y \quad (14b)$$
$$\Delta := \det\{\Omega\}, \quad (14c)$$

the state estimate

$$\hat{x} = \phi^{\text{L}}(\xi + \Phi\hat{\theta}, y), \quad (15)$$

ensures (3) with all signals bounded provided

$$\Delta \notin \mathcal{L}_2. \quad (16)$$

□□□

3.2 An GPEBO with FCT

A variation of GPEBO that ensures FCT is given in Proposition 3. To streamline its presentation we need the following sufficient excitation condition [12]. [3]

Assumption 2 Fix a constant $\mu \in (0,1)$. There exists a time $t_c > 0$ such that

$$\int_0^{t_c} \Delta^2(\tau)d\tau \geq -\frac{1}{\gamma}\ln(1-\mu). \quad (17)$$

Proposition 3 Consider the system (1), verifying the conditions (i)-(iii) of Proposition 1. Fix $\gamma > 0$ and $\mu \in (0,1)$. The state observer defined by (13a)-(13e) and the state estimate

$$\hat{x} = \phi^{\text{L}}\Big(\xi + \Phi_\Lambda \frac{1}{1-w_c}[\hat{\theta} - w_c\hat{\theta}(0)], y\Big), \quad (18)$$

with

$$\dot{w} = -\gamma\Delta^2 w,\ w(0) = 1, \quad (19)$$

and w_c defined via the clipping function

$$w_c = \begin{cases} w & \text{if } \ w < 1-\mu \\ 1-\mu & \text{if } \ w \geq 1-\mu, \end{cases},$$

ensures (4) with all signals bounded provided Δ verifies Assumption 2. □□□

4 Discussion

(D1) The GPEBO PDE (10) is a generalization of the PDEs that are imposed in the Kazantzis-Kravaris-Luenberger observer (KKLO), first presented in [9] as an extension to nonlinear systems of Luenberger's observer, and further developed in [1]. Indeed, in KKLO the mapping $\Lambda(u,y)$ is a *constant*, Hurwitz matrix—see [6] for a recent extension to the non-autonomous case where the mapping ϕ depends on time (or the systems input). It also generalizes the PDE required in PEBO where Λ is *equal to zero*.
(D2) As discussed in [21] and Section 2, a drawback of the original PEBO is that it involves the *open-loop* integration (5), which stymies the practical application of PEBO in the presence of noise—see [21, Remark R5]. Due to the presence of Λ in the dynamics of ξ given in (13a), this difficulty is conspicuous by its absence in GPEBO. It should be pointed out that, using an alternative technique that relies on the Swapping Lemma [28, Lemma 3.6.5], this shortcoming of PEBO has been overcome in [26] for a class of electromechanical systems.
(D3) It is interesting to compare the KKLO with PEBO from the *geometric* viewpoint. The former generates an *attractive and invariant manifold*

$$\mathcal{M} := \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi = \phi(x)\},$$

and the state is reconstructed, via ϕ^{L}, with ξ. On the other hand, PEBO generates an *invariant foliation*

$$\mathcal{M}_\theta := \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n | \xi = \phi(x) + \theta, \theta \in \mathbb{R}^n\},$$

that is, the sublevel sets of the function $F(\xi,x) := \xi - \phi(x)$. To reconstruct the state—again via ϕ^{L}—it is necessary to identify the leaf of $\mathcal{M}_\theta$ via the estimation of θ. See Fig. 1. See also [30] where it is proposed to combine PEBO and KKLO to extend the realm of application of these observers.
(D4) Imposing the algebraic constraint (ii) of Proposition 1 is, clearly, a strong assumption. It is interesting that—as shown in Section 4—it is satisfied for the, practically relevant, power systems example. See also [26] where similar constraints are shown to be satisfied by a class of electromechanical systems and [25] where a significant extension, to the case of *adaptive* state observers—that is, systems with uncertain parameters and unmeasurable states—is reported.
(D5) The version of DREM utilized in Proposition 1

[3] This condition may be defined taking an initial time $t_0 > 0$ and integrating to $t_0 + t_c$. Since we have fixed the initial time everywhere at zero we believe it is more appropriate to leave it like that.

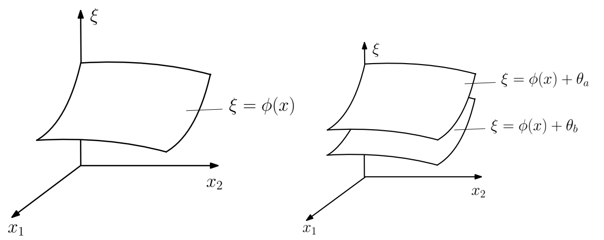

Fig. 1. Geometric interpretation of KKLO and PEBO

uses the dynamic extension proposed by [11]. As discussed in [18] other versions of DREM, with different convergence properties, are also possible. We have opted for this variation for the sake of simplicity.

(D6) The conditions $\Delta \notin \mathcal{L}_2$ and Assumption 2 are, evidently, excitation conditions necessary to ensure convergence of the parameter estimators. Clearly, this kind of assumptions are unavoidable in the problem of state (or parameter) estimation. It is interesting that, as shown in [18], these conditions are *strictly weaker* than the usual persistent of excitation assumption imposed in standard parameter estimation schemes [28, Theorem 2.5.1].

(D7) It is possible to obviate the parameter estimation step of PEBO designing a KKLO-like observer. Indeed, under assumptions (i)-(iii) of Proposition 1 the observer of ϕ given by

$$\dot{\hat{\phi}} = \Lambda(u,y)\hat{\phi} + B(u,y) + \gamma L^\top(u,y)[C(u,y) - L(u,y)\hat{\phi}],$$

verifies the error model

$$\dot{\tilde{\phi}} = [\Lambda(u,y) - \gamma L^\top(u,y)L(u,y)]\tilde{\phi}.$$

where $\tilde{\phi} := \hat{\phi} - \phi$. However, some additional assumptions have to be imposed to the mappings Λ and L to ensure asymptotic stability of this LTV system.

5 Applications

In this section we illustrate with two physical systems the applicability of the proposed GPEBO. Towards this end, we identify all the mappings required to verify some (or all) of the conditions of Proposition 1. For additional details of these examples see the full version of the paper [20].

5.1 Multimachine power systems model

The dynamical model of the i–th generator of n interconnected machines can be described using the classical third order model [4] [14,29]

$$\begin{aligned} \dot{\delta}_i &= \omega_i \\ M_i\dot{\omega}_i &= -D_{mi}\omega_i + \omega_0(P_{mi} - P_{ei}) \\ \tau_i\dot{E}_i &= -E_i - (x_{di} - x'_{di})I_{di} + E_{fi} + \nu_i, \\ & i \in \bar{n} := \{1,...,n\}, \end{aligned} \tag{20}$$

[4] To simplify the notation, whenever clear from the context, the qualifier "$i \in \bar{n}$" will be omitted in the sequel.

where the state variables are the rotor angle $\delta_i \in \mathbb{R}$, rad, the speed deviation $\omega_i \in \mathbb{R}$ in rad/sec and the generator quadrature internal voltage $E_i \in \mathbb{R}_+$, I_{di} is the d axis current, P_{ei} is the electromagnetic power, the voltages E_{fi} and ν_i are the constant voltage component applied to the field winding, and the control voltage input, respectively. D_{mi}, M_i, P_{mi}, τ_i, ω_0, x_{di} and x'_{di} are positive parameters.
The active power P_{ei} and reactive power Q_{ei} are defined as

$$P_{ei} = E_i I_{qi},\ Q_{ei} = E_i I_{di}, \tag{21}$$

where I_{qi} is the q axis current.
These currents establish the connections between the machines and are given by

$$\begin{aligned} I_{qi} &= G_{mii}E_i + \sum_{j=1,j\neq i}^{n} E_j Y_{ij}\sin(\delta_{ij} + \alpha_{ij}) \\ I_{di} &= -B_{mii}E_i - \sum_{j=1,j\neq i}^{n} E_j Y_{ij}\cos(\delta_{ij} + \alpha_{ij}), \end{aligned} \tag{22}$$

where we defined $\delta_{ij} := \delta_i - \delta_j$ and the constants $Y_{ij} = Y_{ji}$ and $\alpha_{ij} = \alpha_{ji}$ are the admittance magnitude and admittance angle of the power line connecting nodes i and j, respectively. Furthermore, G_{mii} is the shunt conductance and B_{mii} the shunt susceptance at node i. Finally, combining (20), (21) and (22) results in the well-known compact form

$$\begin{aligned} \dot{\delta}_i &= \omega_i \\ \dot{\omega}_i &= -D_i\omega_i + P_i - d_iE_i\Big[G_{mii}E_i - \sum_{j=1,j\neq i}^{n} E_jY_{ij}\sin(\delta_{ij} + \alpha_{ij})\Big] \\ \dot{E}_i &= -a_iE_i + b_i\sum_{j=1,j\neq i}^{n} E_jY_{ij}\cos(\delta_{ij} + \alpha_{ij}) + u_i, \end{aligned} \tag{23}$$

where we have defined the new input signal

$$u_i := \frac{1}{\tau_i}(E_{f_i} + \nu_i)$$

and the positive constants

$$\begin{aligned} D_i &:= \frac{D_{mi}}{M_i},\ P_i := d_iP_{mi},\ d_i := \frac{\omega_0}{M_i} \\ a_i &:= \frac{1}{\tau_i}[1 - (x_{di} - x'_{di})B_{mii}],\ b_i := \frac{1}{\tau_i}(x_{di} - x'_{di}). \end{aligned}$$

To formulate the observer problem we consider that all parameters are known, and make the following assumption on the available *measurements*.

Assumption 4 *The signals u_i, δ_i, P_{ei} and Q_{ei} of all generating units are measurable.*

It is fair to say that the assumption of knowledge of δ_i is far from realistic.

5.1.1 *Verifying the conditions of Proposition 1*

We make the following observation. Using (21) and (22), the rotor speed dynamics (23) may be written as

$$\dot{\omega}_i = -D_i\omega_i + P_i - d_iP_{ei}.$$

Considering that P_{ei} is measurable, while P_i, D_i and d_i are known positive constants, the design of an observer for this system is trivial. For instance,

$$\begin{aligned} \dot{\xi}_{\omega_i} &= -D_i\hat{\omega}_i + P_i - d_iP_{ei} - k_{\omega i}\hat{\omega}_i \\ \hat{\omega}_i &= \xi_{\omega_i} + k_{\omega i}\delta_i,\ k_{\omega i} > 0, \end{aligned} \tag{24}$$

yields the LTI, asymptotically stable error dynamics

$$\dot{\tilde{\omega}}_i = -(D_i + k_{\omega i})\tilde{\omega}_i.$$

Therefore, we concentrate in the estimation of the voltages E_i. Its dynamics may be written as

$$\dot{E} = \Lambda(\delta)E + u. \tag{25}$$

where $E := \mathrm{col}(E_1, \ldots, E_n)$, $\delta := \mathrm{col}(\delta_1, \ldots, \delta_n)$, and we defined matrix

$$\Lambda(\delta) := (\Lambda_1(\delta)\ \Lambda_2(\delta)\ \ldots\ \Lambda_n(\delta)), \tag{26}$$

where

$$\Lambda_1(\delta) := \begin{bmatrix} -a_1 \\ b_2Y_{21}\cos(\delta_{21} + \alpha_{21}) \\ b_nY_{n1}\cos(\delta_{n1} + \alpha_{n1}) \end{bmatrix},$$

$$\Lambda_2(\delta) := \begin{bmatrix} b_1Y_{12}\cos(\delta_{12} + \alpha_{12}) \\ -a_2 \\ b_nY_{n2}\cos(\delta_{n2} + \alpha_{n2}) \end{bmatrix},$$

$$\Lambda_n(\delta) := \begin{bmatrix} b_1Y_{1n}\cos(\delta_{1n} + \alpha_{1n}) \\ b_2Y_{2n}\cos(\delta_{2n} + \alpha_{2n}) \\ -a_n \end{bmatrix}$$

and we recall that δ is *measurable*. The remaining mappings of (i) and (ii) of Proposition 1 are given as $\phi = E$ and $B = u$. The following simple lemma defines the mappings L and C that satisfy (12).

Lemma 5 There exists a *measurable* matrix $L(P_e, Q_e, \delta) \in \mathbb{R}^{n\times n}$ such that

$$LE = 0. \tag{27}$$

Consequently, selecting $C = 0$, (12) is satisfied

PROOF. From (21) we have that

$$P_eI_d - Q_eI_q = 0. \tag{28}$$

Clearly, the equations (22)—which are linearly dependent on E—may be written in the compact form

$$I_q = S(\delta)E,\ I_d = T(\delta)E, \tag{29}$$

for some suitably defined $n \times n$ matrices $S(\delta)$, $T(\delta)$. The proof is completed by replacing (29) in (28) and defining

$$L(P_e, Q_e, \delta) := \begin{bmatrix} P_{e1}T_1^\top(\delta) - Q_{e1}S_1^\top(\delta) \\ \vdots \\ P_{en}T_n^\top(\delta) - Q_{en}S_n^\top(\delta) \end{bmatrix},$$

where $T_i^\top(\delta)$, $S_i^\top(\delta)$ are the rows of the matrices $T(\delta)$ and $S(\delta)$, respectively.

This lemma completes the verification of all the conditions of Proposition 1.

5.1.2 *Simulations*

For simulation we use the two-machine system considered in [22]. The dynamics of the system result in the sixth-order model

$$\begin{cases} \dot{\delta}_1 &= \omega_1, \\ \dot{\omega}_1 &= -D_1\omega_1 + P_1 - G_{11}E_1^2 - Y_{12}E_1E_2\sin(\delta_{12} + \alpha_{12}) \\ \dot{E}_1 &= -a_1E_1 + b_1E_2\cos(\delta_{12} + \alpha_{12}) + E_{f_1} + \nu_1; \\ \dot{\delta}_2 &= \omega_2, \\ \dot{\omega}_2 &= -D_2\omega_2 + P_2 - G_{22}E_2^2 + Y_{21}E_1E_2\sin(\delta_{12} + \alpha_{12}) \\ \dot{E}_2 &= -a_2E_2 + b_2E_1\cos(\delta_{21} + \alpha_{21}) + E_{f_2} + \nu_2, \end{cases} \tag{30}$$

with the current equations defined as

$$\begin{aligned} I_{q1} &= G_{11}E_1 + E_2Y_{12}\sin(\delta_{12} + \alpha_{12}) \\ I_{d1} &= -B_{11}E_1 - E_2Y_{12}\cos(\delta_{12} + \alpha_{12}) \\ I_{q2} &= G_{22}E_2 + E_1Y_{21}\sin(\delta_{21} + \alpha_{21}) \\ I_{d2} &= -B_{22}E_2 - E_1Y_{21}\cos(\delta_{21} + \alpha_{21}). \end{aligned}$$

In this case we have that

$$A(t) = \begin{bmatrix} -a_1 & b_1\cos(\delta_{12}(t) + \alpha_{12}) \\ b_2\cos(\delta_{21}(t) + \alpha_{21}) & -a_2 \end{bmatrix}$$

$$S(\delta) = \begin{bmatrix} G_{11} & Y_{12}\sin(\delta_{12} + \alpha_{12}) \\ Y_{21}\sin(\delta_{21} + \alpha_{21}) & G_{22} \end{bmatrix}$$

$$T(\delta) = \begin{bmatrix} -B_{11} & -Y_{12}\cos(\delta_{12} + \alpha_{12}) \\ -Y_{21}\cos(\delta_{21} + \alpha_{21}) & -B_{22} \end{bmatrix}.$$

For the observer design we selected the simplest filter

$$F(p) = \begin{bmatrix} 1 & 0 \\ \frac{k}{\mathbf{p}+k} & 0 \end{bmatrix},$$

with $\mathbf{p} := \frac{d}{dt}$ and $k > 0$. The parameters of the model (30) are taken from [22] and are given in the Table given in Appendix C of [20].

Simulation results are presented in Figs. 2-5. Fig. 2 and Fig. 3 show the observation errors for the open loop observer (OLO) (13a), and for DREM for different adaptation gains and for FCT-DREM. For the simulations we used $\lambda = 1$ in (13c) and (13d). Simulation results for FCT-DREM are presented for $\gamma = 10^7$ in (13e) and $\mu = 0.1$ for the computation ω_c in (18). To test the robustness of the design a 30% load change was introduced at $t = 10$ sec, whose effect is impercebtible. Fig. 4 and Fig. 5 show the observation errors for rotor speed observer (24) for first and second generator for different values of $k_{\omega i}$ in (24).

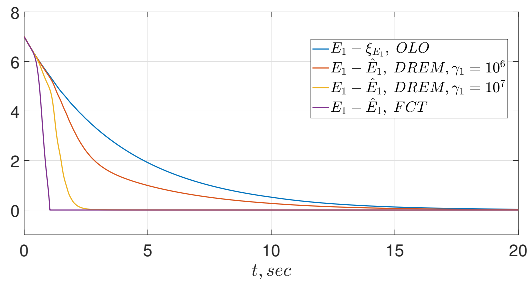

Fig. 2. Transients of the first voltage observation error for the OLO (13a), DREM and FCT-DREM with a 30% load change a $t = 10$ sec

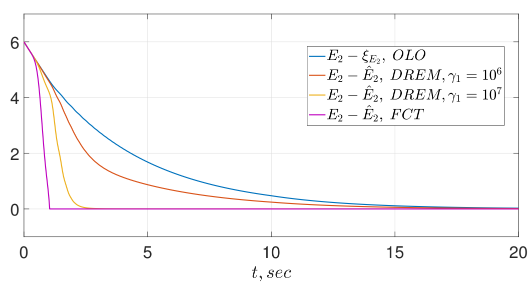

Fig. 3. Transients of the second voltage observation error for the OLO (13a), DREM and FCT-DREM observers with a 30% load change at $t = 10$ sec

5.2 *Chemical-biological reactors*

We consider reaction systems whose dynamical model is given by [4, Section 1.5]

$$\begin{aligned} \dot{c} &= -uc + Kr(c) + \chi \\ y &= \left[I_p \;\vdots\; 0_{p\times d} \right] c, \end{aligned} \tag{31}$$

with $c \in \mathbb{R}^n_+, \chi \in \mathbb{R}^n_+$, $u \in \mathbb{R}_+$, $y \in \mathbb{R}^p$, $r : \mathbb{R}^n \to \mathbb{R}^q_+$, $d := n - p$, $q < n$. It is assumed that y, u, χ and K are

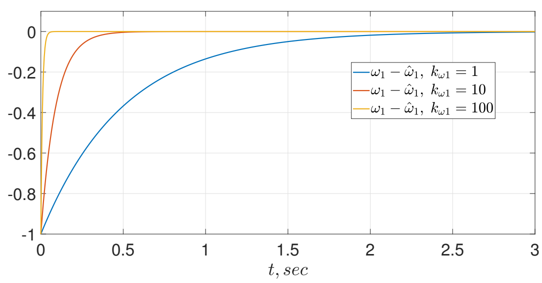

Fig. 4. Transients of the first speed observation error for the observer (24) for different values of $k_{\omega 1}$

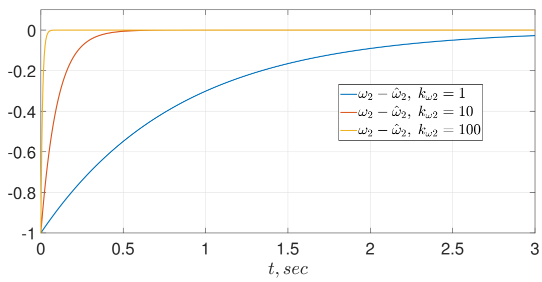

Fig. 5. Transients of the second speed observation error for the observer (24) for different values of $k_{\omega 2}$

known.
To simplify the notation we partition the vector c as $c = \mathrm{col}(y, x)$, and rewrite (31) as

$$\begin{aligned} \dot{y} &= -uy + K_y r(y, x) + \chi_y \\ \dot{x} &= -ux + K_x r(y, x) + \chi_x. \end{aligned} \tag{32}$$

To simplify the presentation we assume that there are *more* measurements than reaction rates, that is, $p \geq q$ and rank $\{K_y\} = q$. [5]

5.2.1 *Solution via GPEBO*

The following lemma identifies the mappings ϕ, Λ and B required to satisfy conditions (i) and (ii) of Proposition 1.

Lemma 6 Consider the system (32). The mappings

$$\begin{aligned} \phi &:= x - K_x K_y^\dagger y \\ \Lambda &:= -u \\ B &:= -K_x K_y^\dagger \chi_y + \chi_x \end{aligned} \tag{33}$$

where
$$K_y^\dagger := (K_y^\top K_y)^{-1} K_y^\top,$$
satisfy the PDE (10). More precisely,

$$\dot{\phi} = \Lambda \phi + B. \tag{34}$$

[5] See [19] for a relaxation of this assumption.

PROOF. From (32) and (33) we get

$$\begin{aligned}\dot{\phi} &= -ux + K_x r(y,x) + \chi_x \\ &\quad - K_x K_y^\dagger[-uy + K_y r(y,x) + \chi_y] \\ &= -u\phi + \chi_x - K_x K_y^\dagger \chi_y,\end{aligned}$$

completing the proof.

Now, note that from (13a), (13b) and (34) we can, invoking the arguments used in the proof of Proposition 1, establish the relation

$$\phi = \xi + \Phi_\Lambda \theta, \tag{35}$$

for some $\theta \in \mathbb{R}^d$. To obtain a *bona fide* regressor equation, that is a linear relation between measurable signals and θ we would assume condition (iii) of Proposition 1. That is, assume the existence of measurable mappings C and L such that (12) holds, that is $L\phi = C$. Unfortunately, in this example it is not possible to satisfy this condition. However, we can still obtain the required linear regression, needed for the parameter estimation using DREM, as shown in the lemma below.

Lemma 7 Assume that the rate vector $r(y,x)$ depends *linearly* on the unmeasurable components of the state x, that is, it is of the form

$$r(y,x) = R(y)x \tag{36}$$

where $R : \mathbb{R}^p \to \mathbb{R}^{q\times d}$ is a known matrix.[6] There exists *measurable* signals $\mathcal{Y} \in \mathbb{R}^d$ and $\Delta \in \mathbb{R}$ such that

$$\mathcal{Y} = \Delta\theta. \tag{37}$$

PROOF. Defining the partial coordinate $y^\dagger = K_y^\dagger y$, we see from (32) that its dynamics takes the form

$$\begin{aligned}\dot{y}^\dagger &= -uy^\dagger + R(y)x + K_y^\dagger \chi_y \\ &= -uy^\dagger + K_y^\dagger\chi_y + R(y)(\phi + K_x y^\dagger) \\ &= -uy^\dagger + K_y^\dagger\chi_y + R(y)(\xi + \Phi_\Lambda\theta + K_x y^\dagger) \\ &= \Psi\theta + \chi_l\end{aligned} \tag{38}$$

where we used (33) to get the second identity, (35) in the third identity and we defined the measurable signals

$$\begin{aligned}\chi_l &:= -uy^\dagger + K_y^\dagger\chi_y + R(y)(\xi + K_x y^\dagger) \\ \Psi &:= R(y)\Phi_\Lambda.\end{aligned}$$

Applying the filter $\frac{\lambda}{\mathbf{p}+\lambda}$—with $\lambda > 0$ a *free* tuning parameter—to (38), and regrouping terms, we obtain the linear regression equation[7]

$$Y = \Psi_f \theta. \tag{39}$$

where we defined the signals

$$\Psi_f := \frac{\lambda}{\mathbf{p}+\lambda}[\Psi],\ Y := \frac{\lambda\mathbf{p}}{\mathbf{p}+\lambda}[y^\dagger] - \frac{\lambda}{\mathbf{p}+\lambda}[\chi_l]. \tag{40}$$

Multiplying (39) by $\mathrm{adj}\{\Psi_f^\top\Psi_f\}\Psi_f^\top$ we obtain the identity (37), where we defined

$$\mathcal{Y} := \mathrm{adj}\{\Psi_f^\top\Psi_f\}\Psi_f^\top Y,\ \Delta := \det\{\Psi_f^\top\Psi_f\}. \tag{41}$$

This completes the proof.

5.2.2 Simulations

To illustrate the performance of the DREM observer proposed in the previous section we consider the model of the anaerobic digestion reactor reported in [17]. The dynamics, given in equations (55)-(59) of [17], may be written in the form (32), (36) with the choices $n = 4, q = 2, p = 2$

$$K_y = \begin{bmatrix} -k_3 & 0 \\ k_4 & -k_1\end{bmatrix},\ K_x = I_2$$

$$R(y) = \begin{bmatrix}\mu_1(y_1) & 0 \\ 0 & \mu_2(y_2)\end{bmatrix},\ \chi_y = \begin{bmatrix} us_{1,0} \\ us_{2,0}\end{bmatrix},\ \chi_x = 0,$$

where y_1, x_1, y_2 and x_2 represent the organic matter concentration (g/l), the acidogenic bacteria concentration (g/l), the volatile fatty acid concentration (mmol), the methanogenic bacteria concentration (g/l) and u is the dilution rate. The positive constants $s_{1,0}$ and $s_{2,0}$ denote the concentration of the substrate in the feed, and k_1, k_3 and k_4 are yield positive coefficients.
The two specific growth rates μ_1 and μ_2 are given by

$$\begin{bmatrix}\mu_1(y_1) \\ \mu_2(y_2)\end{bmatrix} = \begin{bmatrix}\frac{\mu_{m,1}y_1}{K_{S,1}+y_1} \\ \frac{\mu_{m,2}y_2}{K_{S,2}+y_2+K_I y_2^2}\end{bmatrix}.$$

where $\mu_{m,1}$, $\mu_{m,2}$, $K_{S,1}$, $K_{S,2}$ and K_I are yield positive coefficients.
Notice that K_y is square and full rank, consequently

$$y^\dagger = K_y^{-1}y = -\begin{bmatrix}\frac{y_1}{k_3} \\ \frac{y_2}{k_1} + \frac{k_4 y_1}{k_1 k_3}\end{bmatrix}.$$

[6] See [19] for the case of nonlinear dependence on x.

[7] As usual in adaptive control, we neglect an additive exponentially decaying term in (39) that is due to the filters initial conditions.

To design the observer we first identify the signals (33) of Lemma 6 as

$$\Lambda = -u$$
$$B = -K_y^{-1}\chi_y = -u\begin{bmatrix} \frac{-s_{1,0}}{k_3} \\ \frac{-s_{2,0}}{k_1} - \frac{k_4 s_{1,0}}{k_1 k_3}\end{bmatrix}.$$

Consequently, (13a) and (13b) become

$$\dot{\xi} = -u\xi + u\begin{bmatrix} \frac{s_{1,0}}{k_3} \\ \frac{s_{2,0}}{k_1} + \frac{k_4 s_{1,0}}{k_1 k_3}\end{bmatrix}$$
$$\dot{\Phi}_\Lambda = -u\Phi_\Lambda,\ \Phi_\Lambda(0) = I_n.$$

Then, we follow the proof of Lemma 7 to construct the signals

$$\chi_l = u\begin{bmatrix} \frac{y_1}{k_3} \\ \frac{y_2}{k_1} + \frac{k_4 y_1}{k_1 k_3}\end{bmatrix} - u\begin{bmatrix} \frac{s_{1,0}}{k_3} \\ \frac{s_{2,0}}{k_1} + \frac{k_4 s_{1,0}}{k_1 k_3}\end{bmatrix} + \begin{bmatrix} \mu_1(y_1)[\xi_1 - \frac{y_1}{k_3}] \\ \mu_2(y_2)[\xi_2 - \frac{y_2}{k_1} - \frac{k_4 y_1}{k_1 k_3}]\end{bmatrix}$$
$$\Psi = \begin{bmatrix} \mu_1(y_1) & 0 \\ 0 & \mu_2(y_2)\end{bmatrix}\Phi,$$

that, together with (40) and (41), define $\mathcal{Y}$ and Δ of (37). The design is completed with the parameter estimator (13e).

For the simulations we used the parameters of [17], that is, $k_1 = 268$ mmol/g, $k_3 = 42.14$, $k_4 = 116.5$ mmol/g, $\alpha = 1$,[8] $\mu_{m,1} = 1.2\ d^{-1}$, $K_{S,1} = 8.85$ g/l, $\mu_{m,2} = 0.74\ d^{-1}$, $K_{S,2} = 23.2$ mmol, $K_I = 0.0039$ mmol^{-1}, $S_{1,0} = 1$, $S_{2,0} = 1$ and $u = 0.1$.

The initial conditions for the anaerobic digester were set to $x_1(0) = 0.1$ g/l, $y_1(0) = 0.05$ g/l, $x_2(0) = 0.5$ g and $y_2(0) = 4$ mmol/l. We used $\lambda = 100$ in the filters of (40). Fig. 6 and Fig. 7 show the transient behavior of the state estimation errors for different values of the adaptation gain, with $\gamma = 0$ corresponding to the OLO. Notice that, although the convergence rate is increased with larger γ, an undesirable peak appears at the beginning of the error transient.

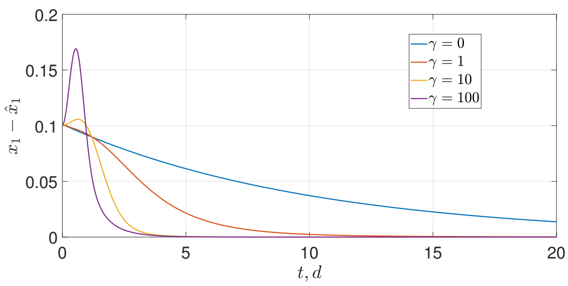

Fig. 6. Transients of the error $x_1 - \hat{x}_1$

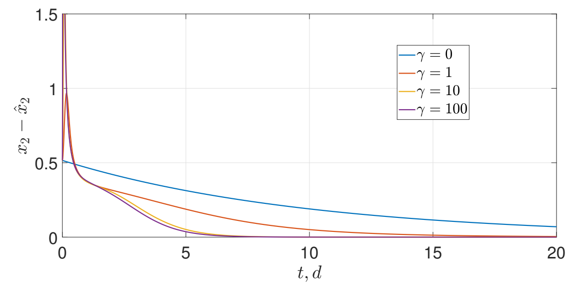

Fig. 7. Transients of the error $x_2 - \hat{x}_2$

6 Concluding Remarks

An extension to the PEBO technique reported in [21] has been proposed in the paper. It allows us to simplify the task of solving the key PDE and avoid a, sometimes problematic, open-loop integration required in PEBO. Also, we have identified a condition—verification of the algebraic equation (12)—that trivializes the task of estimating the unknown parameters. In the original version of PEBO this was left as an open problem to be solved. It is shown that this condition is satisfied for the practically important problem of power systems.

It has been shown that combining PEBO with DREM it is possible, on one hand, to relax the excitation conditions to ensure parameter convergence. On the other hand, it allows us to design an observer with FCT under weak excitation assumptions.

As an additional example we show the application of PEBO+DREM to reaction systems. Notice that the use of DREM is necessary to solve the parameter estimation problem in this example. Although there are many ways to design an estimator from the linear regression (39), there exists a fundamental obstacle to ensure its convergence. Indeed, from the definition of Φ_Λ, that is $\dot{\Phi}_\Lambda = -u\Phi_\Lambda$ with $u(t) > 0$, we have that $\Phi_\Lambda(t) \to 0$, hence $\Psi(t) \to 0$—loosing identifiability of the parameter θ. In particular the matrix Ψ *cannot satisfy* the well-known persistency of excitation condition

$$\int_t^{t+\kappa} \Psi^\top(s)\Psi(s)ds \geq \kappa I_d,$$

which is the necessary and sufficient condition for exponential convergence of the classical gradient and least-squares estimators [28, Theorem 2.5.1].

[8] In [17] there is a constant $\alpha = 0.5$ entering into the dyamics of x as $\dot{x} = -\alpha u x + K_x r(y,x) + \chi_x$. To avoid cluttering the notation, and without loss of generality, we assume this constant is equal to one.

Acknowledgements

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References

[1] Vincent Andrieu and Laurent Praly. On the existence of a kazantzis-kravaris/luenberger observer. *SIAM Journal on Control and Optimization*, 45(2):432–456, 2006.

[2] Stanislav Aranovskiy, Alexey Bobtsov, Romeo Ortega, and Anton Pyrkin. Performance enhancement of parameter estimators via dynamic regressor extension and mixing. *IEEE Transactions on Automatic Control*, 62(7):3546–3550, 2017.

[3] Alessandro Astolfi, Dimitrios Karagiannis, and Romeo Ortega. *Nonlinear and Adaptive Control with Applications*. Springer Science & Business Media, 2007.

[4] George Bastin and Denis Dochain. *On-line Estimation and Adaptive Control of Bioreactors: Elsevier, Amsterdam, (ISBN 0-444-88430-0)*. Elsevier, 1991.

[5] Pauline Bernard. *Observer Design for Nonlinear Systems*, volume 479. Springer, 2019.

[6] Pauline Bernard and Vincent Andrieu. Luenberger observers for nonautonomous nonlinear aystems. *IEEE Transactions on Automatic Control*, 64(1):270–281, 2018.

[7] Gildas Besançon. *Nonlinear Observers and Applications*, volume 363. Springer, 2007.

[8] Denis Dochain. *Automatic Control of Bioprocesses*. John Wiley & Sons, 2013.

[9] Nikolaos Kazantzis and Costas Kravaris. Nonlinear observer design using lyapunov′s auxiliary theorem. *Systems and Control Letters*, 34(5):241–247, 1998.

[10] Hassan K. Khalil. *High Gain Observers in Nonlinear Feedback Control*. SIAM Publishers, 2017.

[11] Gerhard Kreisselmeier. Adaptive observers with exponential rate of convergence. *IEEE transactions on automatic control*, 22(1):2–8, 1977.

[12] Gerhard Kreisselmeier and Gudrun Rietze-Augst. Richness and excitation on an interval-with application to continuous-time adaptive control. *IEEE transactions on automatic control*, 35(2):165–171, 1990.

[13] Abhinav Kumar and Bikash Pal. *Dynamic Estimation and Control of Power Systems*. Academic Press, 2018.

[14] Prabha Kundur, Neal J. Balu, and Mark G. Lauby. *Power System Stability and Control*, volume 7. McGraw-hill New York, 1994.

[15] Ljung Lennart. *System Identification: Theory for the User*. Prentice Hall, New Jersey, USA, 1987.

[16] David Luenberger. Observers for multivariable systems. *IEEE Transactions on Automatic Control*, 11(2):190–197, 1966.

[17] Nikolas Marcos, Martin Guay, Denis Dochain, and Tao Zhang. Adaptive extremum-seeking control of a continuous stirred tank bioreactor with haldane's kinetics. *Journal of Process Control*, 14(3):317–328, 2004.

[18] Romeo Ortega, Stanislav Aranovskiy, Anton A Pyrkin, Alessandro Astolfi, and Alexey A Bobtsov. New results on parameter estimation via dynamic regressor extension and mixing: continuous and discrete-time cases. *arXiv preprint arXiv:1908.05125*, 2019.

[19] Romeo Ortega, Alexey Bobtsov, Denis Dochain, and Nikolay Nikolaev. State observers for reaction systems with improved convergence rates. *Journal of Process Control*, 83:53–62, 2019.

[20] Romeo Ortega, Alexey Bobtsov, Nikolay Nikolaev, Johannes Schiffer, and Denis Dochain. Generalized parameter estimation-based observers: application to power systems and chemical-biological reactors. *arXiv:2003.10952*, 2020.

[21] Romeo Ortega, Alexey Bobtsov, Anton Pyrkin, and Stanislav Aranovskiy. A parameter estimation approach to state observation of nonlinear systems. *Systems & Control Letters*, 85:84–94, 2015.

[22] Romeo Ortega, Martha Galaz, Alessandro Astolfi, Yuanzhang Sun, and Tielong Shen. Transient stabilization of multimachine power systems with nontrivial transfer conductances. *IEEE Transactions on Automatic Control*, 50(1):60–75, 2005.

[23] Romeo Ortega, Dmitry N. Gerasimov, Nikita E. Barabanov, and Vladimir O. Nikiforov. Adaptive control of linear multivariable systems using dynamic regressor extension and mixing estimators: removing the high-frequency gain assumptions. *Automatica*, 2019.

[24] Romeo Ortega, Laurent Praly, Stanislav Aranovskiy, Bowen Yi, and Weidong Zhang. On dynamic regressor extension and mixing parameter estimators: Two luenberger observers interpretations. *Automatica*, 95:548–551, 2018.

[25] Anton Pyrkin, Alexey Bobtsov, Romeo Ortega, Alexei Vedyakov, and Stanislas Aranovskiy. Adaptive state observer design using dynamic regressor extension and mixing. *Systems and Control Letters*, 133:1–8, 2019.

[26] Anton A. Pyrkin, Alexey A. Vedyakov, Romeo Ortega, and Alexey A. Bobtsov. A robust adaptive flux observer for a class of electromechanical systems. *International Journal of Control*, pages 1–11, 2018.

[27] Wilson Rugh. *Linear Systems Theory*. Prentice Hall, NJ, USA, 1996.

[28] Shankar Sastry and Marc Bodson. *Adaptive Control: Stability, Convergence and Robustness*. Prentice Hall, New Jersey, USA, 1989.

[29] Peter W. Sauer, Mangalore A. Pai, and Joe H. Chow. *Power System Dynamics and Stability: with Synchrophasor Measurement and Power System Toolbox*. John Wiley & Sons, 2017.

[30] Bowen Yi, Romeo Ortega, and Weidong Zhang. On state observers for nonlinear systems: A new design and a unifying framework. *IEEE Transactions on Automatic Control*, 64(3):1193–1200, 2018.

A Proof of Proposition 1

From (10) we have that

$$\dot{\phi} = \Lambda\phi + B.$$

Hence, defining the error signal

$$e := \phi - \xi \tag{A.1}$$

and taking into account the ξ dynamics of the observer, we obtain an LTV system $\dot{e} = A(t)e$ where we defined $A(t) := \Lambda(u(t), y(t))$. Now, from the (13b) we see that

Φ is the *fundamental matrix* of the e system, which is bounded in view of condition (iv). Consequently, there exists a *constant* vector $\theta \in \mathbb{R}^n$ such that

$$e = \Phi\theta,$$

namely $\theta = e(0)$. We now have the following chain of implications

$$\begin{aligned} e = \Phi\theta \Leftrightarrow\ & \phi = \xi + \Phi\theta \quad (\Leftarrow (A.1)) \\ \Rightarrow\ & L\phi = L\xi + L\Phi\theta \quad (\Leftarrow L\times) \\ \Rightarrow\ & C - L\xi = L\Phi\theta \quad (\Leftarrow (12)) \\ \Leftrightarrow\ & C - L\xi = \Psi\theta \quad (\Leftarrow (14a)) \\ \Rightarrow\ & \Psi^\top(C - L\xi) = \Psi^\top\Psi\theta \quad (\Leftarrow \Psi^\top\times) \\ \Rightarrow\ & Y = \Omega\theta \quad \Big(\Leftarrow \frac{\lambda}{\mathbf{p}+\lambda}[\cdot] \text{ and } (13c), (13d)\Big) \\ \Rightarrow\ & \Delta\theta = \mathcal{Y}, \quad (\Leftarrow \text{adj}\{\Omega\}\times \text{ and } (14b), (14c)) \end{aligned}$$

where we have used the fact that for any, *possibly singular*, $n \times n$ matrix K we have $\text{adj}\{K\}K = \det\{K\}I_n$ in the last line.

From $\phi = \xi + \Phi\theta$ and (11) it is clear that, if θ is known, we have that

$$x = \phi^{\mathsf{L}}(\xi + \Phi\theta, y). \tag{A.2}$$

Hence, the remaining task is to generate an *estimate* for θ, denoted $\hat{\theta}$, to obtain the observed state via $\hat{x} = \phi^{\mathsf{L}}(\xi + \Phi\hat{\theta}, y)$. This is, precisely, generated with (13e), whose error equation is of the form

$$\dot{\tilde{\theta}} = -\gamma\Delta^2\tilde{\theta}, \tag{A.3}$$

where $\tilde{\theta} := \hat{\theta} - \theta$. The solution of this equation is given by

$$\tilde{\theta}(t) = e^{-\gamma\int_0^t \Delta^2(s)ds}\tilde{\theta}(0). \tag{A.4}$$

Given the standing assumption on Δ we have that $\tilde{\theta}(t) \to 0$. Hence, invoking (15) and (A.2) we conclude that $\tilde{x}(t) \to 0$, where $\tilde{x} := \hat{x} - x$.

B Proof of Proposition 3

First, notice that the definition of w_c ensures that $\hat{x}$, given in (18), is well-defined. Now, from (A.4) and the definition of w we have that

$$\tilde{\theta} = w\tilde{\theta}(0).$$

Clearly, this is equivalent to

$$(1 - w)\theta = \hat{\theta} - w\hat{\theta}(0).$$

On the other hand, under Assumption 2, we have that $w_c(t) = w(t)$, $\forall t \geq t_c$. Consequently, we conclude that

$$\frac{1}{1 - w_c}[\hat{\theta} - w_c\hat{\theta}(0)] = \theta, \ \forall t \geq t_c.$$

Replacing this identity in (18) completes the proof.