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Strong Lyapunov functions for two classical problems in adaptive control[☆]

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ABSTRACT

Strong Lyapunov functions for two classical problems in adaptive control and parameter identification are presented. These Lyapunov functions incorporate in their structure the classical persistency of excitation conditions, allowing to show global uniform asymptotic stability of the associated adaptive systems under sufficient and necessary conditions.

1. Introduction

Classical adaptive control deals with the identification of unknown and constant parameters. A good part of the classical problems can be studied through the following two Linear Time-Varying (LTV) systems (Narendra & Annaswamy, 1989, Sec. 2.8)

$$\dot{x}(t) = -\Gamma C^{\top}(t)C(t)x(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $C(t) \in \mathbb{R}^{m\times n}$ is the *regressor* and $\Gamma \in \mathbb{R}^{n\times n}$ is a design gain, and

$$\dot{z}(t) = \mathcal{A}(t)z(t),\quad \mathcal{A}(t) = \begin{bmatrix} A(t) & B(t) \\ -B^{\top}(t)P(t) & 0 \end{bmatrix}, \tag{2}$$

where $z(t) \in \mathbb{R}^{n+m}$ is the state, $B(t) \in \mathbb{R}^{m\times n}$ is the *regressor* and $A(t), P(t) \in \mathbb{R}^{m\times m}$ are given known matrices. The regressors $C(\cdot)$ and $B(\cdot)$ are *piecewise continuous* functions. The first system, system (1), is the error dynamics of a parameter estimation process (Anderson, 1977; Morgan & Narendra, 1977b; Narendra & Annaswamy, 1989), whereas system (2) appears, for example, when a linear system is subjected to an adaptive control or as the error dynamics of an adaptive observer (Anderson, 1977; Morgan & Narendra, 1977a; Narendra & Annaswamy, 1989). In such situation, $A(t)$ and $P(t)$ represent the gains of the adaptive control/observer and are specified by the designer.

Necessary and sufficient conditions for the Global Uniform Asymptotic Stability (GUAS) of the zero equilibrium solution of (1) and (2) have been obtained in Anderson (1977) and Morgan and Narendra (1977a, 1977b). The proofs rely on *weak* Lyapunov functions (LFs), i.e., LFs having only negative semi-definite derivatives, and some geometric methods for Morgan and Narendra (1977a, 1977b), and using the connections between uniform complete observability (UCO) and GUAS in Anderson (1977). Although for LTV systems GUAS (or equivalently Uniform Exponential Stability) implies the existence of a *strong* quadratic Lyapunov function (Khalil, 2002, Thm. 4.12) and (Anderson, 1977, Lem. 2), i.e., a LF having negative definite derivative, such functions have not been yet given explicitly for systems (1) and (2) under the most general GUAS conditions for non-smooth regressors.

Recently, some explicit *strong* LFs for (1) and (2) have been found (see e.g. Aranovskiy, Ortega, Romero, & Sokolov, 2019; Loría, Panteley, & Maghenem, 2019a; Loría, Panteley, & Maghenem, 2019b; Maghenem & Loría, 2017; Maghenem & Loría, 2017; Mazenc, de Queiroz, & Malisoff, 2009), but under conditions much more restrictive than those required by the systems to be GUAS. The objective (and novelty) of this note is to exhibit, *explicit*, smooth and quadratic strong Lyapunov functions for systems (1) and (2) under *necessary and sufficient* conditions for GUAS, i.e., the origin of these systems is GUAS iff the corresponding function is a strong LF. It is well-known that having strong LFs is advantageous for e.g. robust analysis, study of input/output properties as Input-to-State Stability (ISS), calculation of convergence velocity, etc.

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The note structure is as follows: In Section 2, the necessary and sufficient conditions for GUAS of systems (1) and (2) are recalled; this work contribution is also given in this section in item *(iii)* of Theorems 1 and 3. In Section 3, the proposed LFs are discussed. In Section 4, the results available in the literature are reviewed and contrasted with the proposed ones; finally, in Section 5, the proof of the main result of this work is given.

Notation. Along the note, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ the real n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ the set of real $n\times m$ matrices. $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. For $A, B \in \mathbb{R}^{n\times n}$ symmetric, $A > B$ ($A \geq B$) means that $A-B$ is positive (semi) definite. For $v \in \mathbb{R}^n$, $\|v\|$ denotes $= (v^\top v)^{1/2}$ and for $B \in \mathbb{R}^{m\times n}$, $\|B\|$ denotes the induced norm of B, defined as $\sup_{\|x\|=1} \|Bx\|$. For a $A = A^\top$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues of A. The function space $PC\big([0,\ \infty),\ \mathbb{R}^{n\times m}\big)$ is the set of all functions mapping non negative real values into $\mathbb{R}^{n\times m}$ which are *piecewise continuous*, i.e., they are continuous everywhere, except that they may have a *finite* number of discontinuity points on every bounded subinterval, where the one-sided limits are well defined and finite. Moreover, $R(t) \in PC^1\big([0,\ \infty),\ \mathbb{R}^{n\times m}\big)$ if $\dot{R}(t)$ exists almost everywhere and $\dot{R}(t) \in PC\big([0,\ \infty),\ \mathbb{R}^{n\times m}\big)$.

2. Strong Lyapunov functions for the two classical systems in adaptive control

Here the (classical) conditions for systems (1) and (2) to be GUAS are recalled and the proposed strong LFs are presented. The proofs are given in Section 5.

Theorem 1. *Let Γ in (1) be symmetric and positive definite, with $\|\Gamma\| = r_1$. Let $C(\cdot) \in PC\big([0,\ \infty),\ \mathbb{R}^{n\times m}\big)$ be bounded, i.e., $\|C(t)\| \leq r_2\ \forall\, t \geq 0$. Then the following statements are equivalent.*

(i) The origin of system (1) is GUAS.
(ii) There exist constants $\gamma_1 \geq \gamma_2 > 0$ and $T > 0$, all independent of t, s.t. for all $t \geq T$

$$\gamma_1 \mathbf{I}_n \geq \int_{t-T}^{t} C^\top(\sigma)C(\sigma)\mathrm{d}\sigma \geq \gamma_2 \mathbf{I}_n\,. \tag{3}$$

(iii) The quadratic function $V(x,\ t) = x^\top \mathcal{P}(t)x$, with

$$\mathcal{P}(t) = \frac{1}{2}\left(\frac{2(T r_1 r_2 \gamma_1)^2}{\gamma_2} + T\right)\Gamma^{-1} + \int_{t-T}^{t} (\sigma - t + T)C^\top(\sigma)C(\sigma)\mathrm{d}\sigma \tag{4}$$

differentiable, is a strong Lyapunov function.

Condition (3) is known in the literature as *Persistency of Excitation (PE)*. It is equivalent to *Uniform Complete Observability (UCO)* of the system

$$\dot{\theta}(t) = 0, \quad y(t) = C(t)\theta(t)\,. \tag{5}$$

This is used for the proof of the equivalence of items (i) and (ii) of Theorem 1 in Anderson (1977, Thm. 1).

Before continuing, we recall the following GUAS stability result for LTV systems (see, e.g., Khalil, 2002, Thm 4.12).

Lemma 2. *Let $F(\cdot) \in PC\big([0,\ \infty),\ \mathbb{R}^{m\times m}\big)$ be bounded, i.e., $\|F(t)\| \leq r_4$ for all $t \geq 0$. Let $Q(\cdot) \in PC\big([0,\ \infty),\ \mathbb{R}^{m\times m}\big)$ be any symmetric, bounded and positive definite matrix, i.e., $\eta_4 \mathbf{I}_m \geq Q(t) \geq \eta_3 \mathbf{I}_m$ for some constants $\eta_4 \geq \eta_3 > 0$. Then the following are equivalent.*

(a) The origin of the LTV system $\dot{x} = F(t)x$ is GUAS.
(b) There exists a symmetric, bounded and positive definite solution $R(\cdot) \in PC^1\big([0,\ \infty),\ \mathbb{R}^{m\times m}\big)$ of the Differential Lyapunov Equation (DLE) (6), i.e., $\eta_1 \mathbf{I}_m \geq R(t) \geq \eta_2 \mathbf{I}_m > 0$ for some constants $\eta_1 \geq \eta_2 > 0$, and

$$\dot{R}(t) + R(t)F(t) + F^\top(t)R(t) = -Q(t)\,. \tag{6}$$

Theorem 3. *Let $B(\cdot) \in PC\big([0,\ \infty),\ \mathbb{R}^{n\times m}\big)$ be bounded, i.e., $\|B(t)\| \leq r_3\ \forall\, t \geq 0$, and $A(\cdot)$ and $P(\cdot)$ in (2) satisfy the conditions of Lemma 2 for $F(t) = A(t)$ and $R(t) = P(t)$ and some $Q(t)$. Then the following statements are equivalent.*

(i) The origin of system (2) is GUAS.
(ii) There exist constants $\gamma_3 \geq \gamma_4 > 0$ and $T > 0$, all independent of t, s.t. for all $t \geq T$

$$\gamma_3 \mathbf{I}_n \geq \int_{t-T}^{t} \mathcal{K}^\top(t,s)\mathcal{K}(t,s)\mathrm{d}s \geq \gamma_4 \mathbf{I}_n\,, \tag{7}$$

$$\mathcal{K}(a,b) := \int_{b}^{a} B(\sigma)\mathrm{d}\sigma\,.$$

(iii) The quadratic function

$$V(z,t) = z^\top \mathcal{P}(t) z = z^\top \Big(k\Pi_1(t) + \Pi_2(t)\Big) z, \tag{8}$$

with $\mathcal{P}(t)$ differentiable and where

$$\Pi_1(t) = \begin{bmatrix} P(t) & 0 \\ 0 & \mathbf{I}_n \end{bmatrix},\ \Pi_2(t) = \begin{bmatrix} 0 & \mathcal{P}_{12}(t) \\ \mathcal{P}_{12}^\top(t) & \mathcal{P}_{22}(t) \end{bmatrix}, \tag{9}$$

$$\mathcal{P}_{12}(t) = -\int_{t-T}^{t} \big(s - t + T\big)\mathcal{K}(t,s)\mathrm{d}s,$$

$$\mathcal{P}_{22}(t) = \int_{t-T}^{t} \big(s - t + T\big)\mathcal{K}^\top(t,s)\mathcal{K}(t,s)\mathrm{d}s\,,$$

is a strong LF for $k > 0$ sufficiently large.

Condition (7) is equivalent to UCO of the system

$$\dot{y}(t) = B(t)\theta(t), \quad \dot{\theta}(t) = 0, \tag{10}$$

with output y. This is used in Anderson (1977, Thm. 2) to show the equivalence of items (i) and (ii) of Theorem 3.

Conditions (3) and (7) differ slightly from the corresponding ones given in Morgan and Narendra (1977b, Thm. 1) and Morgan and Narendra (1977a, Thm. 1), but they are equivalent.

Remark 4. Conditions (3) and (7) are related, but they are in general not equivalent (if we set $C(t) = B(t)$). Proposition 1 in Anderson (1977) and Corollary 2 in Morgan and Narendra (1977a) (see also the example after Corollary 2 in Morgan & Narendra, 1977a) show that (3) is *necessary* for (7), but not sufficient. However, if $B(t)$ is smooth and $|\dot{B}(t)|$ is uniformly bounded, then (3) and (7) are equivalent. Thus, Theorem 3 requires the weakest possible conditions for GUAS of (2).

Remark 5. For simplicity, we restricted the matrix functions to be *PC*. However, broader classes of functions such as regulated or integrable functions can be considered with little to no effort.

3. Discussion of the results

Weak LF for systems (1) and (2) are well-known. The standard weak LF and its derivative for (1) are (Maghenem & Loría, 2017; Narendra & Annaswamy, 1989)

$$V(x) = \frac{1}{2}x^\top \Gamma^{-1} x,\ \dot{V}(t) = -x^\top C^\top(t)C(t)x, \tag{11}$$

while for (2) is Maghenem and Loría (2017) and Narendra and Annaswamy (1989)

$$V(z,t) = z_1^\top P(t) z_1 + z_2^\top z_2,\ \dot{V}(t) = -z_1^\top Q(t) z_1\,. \tag{12}$$

Without extra conditions on the regressors they assure Global Uniform Stability. For GUAS the excitation conditions (3) and (7) are required for (1) and (2), respectively. The works Morgan and Narendra (1977a, 1977b) and Anderson (1977) show this using the weak LF and further arguments.

We note that the strong LF for system (1) proposed in Theorem 1 item (iii) corresponds to the weak LF (11) with an additional strictifying term related to the PE condition (3), i.e., the observability (UCO) condition. Similarly, the strong LF for system (2) proposed in Theorem 3 item (iii) corresponds to the weak LF (12) with additional strictifying terms $\mathcal{P}_{12}(t)$ and $\mathcal{P}_{22}(t)$, related to the persistency excitation condition (7), i.e., the UCO condition. In contrast to the previously proposed strong LFs for both systems (Aranovskiy et al., 2019; Loría et al., 2019a, 2019b; Maghenem & Loría, 2017; Mazenc, de Queiroz et al., 2009), these extra terms depend on the integral of the regressor matrix, and therefore do not require the regressor to be differentiable.

In view of Lemma 2, finding a strong LF for system (1) or (2) amounts to obtaining a solution to the corresponding DLE (6). What is interesting though is to get an *explicit* solution to the DLE. Although this may appear to be a simple task, surprisingly only some proposals have been recently given in the literature (Aranovskiy et al., 2019; Loría et al., 2019a, 2019b; Maghenem & Loría, 2017; Mazenc, de Queiroz et al., 2009), which require extra conditions to the ones given in Theorems 1 and 3. A numerical solution of the DLE is of course possible, but this makes it difficult to study other system's properties. Recently, Praly[1] (Praly, 2019, Example 4) proposes a method to construct a strong LF for system (2). However, it is not *explicit*, since it requires the solution of a (nonlinear) Differential Riccati Equation. In this sense, this method is akin to integrating the DLE (6).

In contrast, the explicit LFs given in item (iii) of Theorems 1 and 3 allow to compute explicit bounds, that can be used to compute the convergence rate or to investigate ISS gains for the systems. For example, the LF $V(x,t)$ given in Theorem 1 satisfy

$$\kappa_1\|x\|^2 \geq V(x,t) \geq \kappa_2\|x\|^2,\ \ \dot{V}(t) \leq -\frac{\gamma_2}{2}\|x(t)\|^2,$$

where the constants κ_1 and κ_2 are given in (15) and the bound over $\dot{V}$ is given at the end of Section 5.1. Using them it is straightforward to show that

$$\|x(t)\| \leq \sqrt{\frac{\kappa_1}{\kappa_2}}\|x(t_0)\| \exp\left(-\frac{\gamma_2}{4\kappa_1}(t-t_0)\right).$$

This yields, for an additive disturbance $\delta(t)$ of the form $\dot{x}(t) = -\Gamma C^\top(t)C(t)x(t)+\delta(t)$, the ISS-gain

$$\gamma_{\mathrm{ISS}} = \frac{4\kappa_1\sqrt{\kappa_1}}{\gamma_2\sqrt{\kappa_2}}.$$

The expression above gives the rate of convergence of system (1), a property that has been investigated in e.g. Brockett (2000) and Loría and Panteley (2002) by more intricate methods, obtaining limited results. This shows the usefulness of explicit strong LFs. Finally, it is worth mentioning that analogous bounds for $V(z,t)$ in item (iii) of Theorem 3 can be derived, yielding the corresponding rate of convergence.

4. Comparison with previous works

For system (1) the most relevant result is given in Maghenem and Loría (2017, Lem. 1). However, it is only obtained for the case $n=1$. The LF in Maghenem and Loría (2017) coincides with (4) for $n=1$.

[1] Praly (2019) appeared after the first submission of our paper.

For system (2) several *explicit* strong LFs have been given in Aranovskiy et al. (2019), Loría et al. (2019a, 2019b), Maghenem and Loría (2017) and Mazenc, de Queiroz et al. (2009). In most of these works a strong LF has been constructed for a more general nonlinear version of the problem. When specialized to the LTV system (2), they all impose the following (common) restrictions, in contrast to the solution proposed in our paper: (R1) The regressor $B(\cdot)$ is bounded and continuously differentiable with a bounded derivative $\dot{B}$. For the LTV systems this restriction prevents the usage of the LFs in the case of piecewise constant or piecewise smooth regressors, two very common classes of signals in adaptive control, or for switched systems. (R2) It has to satisfy condition (3) (not (7)). (R3) The proposed LF includes terms involving directly the regressor $B(\cdot)$, rendering its differentiability necessary. As a consequence, the LFs in Aranovskiy et al. (2019), Loría et al. (2019a, 2019b), Maghenem and Loría (2017) and Mazenc, de Queiroz et al. (2009) cannot be used if the regressor is non-smooth.

To illustrate the implications of the smoothness restriction over $B(t)$, consider system (2) with $n=m=1$, i.e.,

$$\dot{z}_1(t) = -a(t)z_1(t) + b(t)z_2(t),$$
$$\dot{z}_2(t) = -p(t)b(t),$$

with the discontinuous regressor $b(t)$ defined as

$$b(t) = \begin{cases} \sqrt{\tau^2-(t-t_n)^2}+c & \text{if} \quad n \text{ is even}, \\ -\sqrt{\tau^2-(t-t_n)^2}-c & \text{if} \quad n \text{ is odd} \end{cases},$$

for $t \in [t_n-\tau, t_n+\tau)$ and where n is a natural number, τ and c are arbitrary positive constants and the sequence $\{t_n\}$ is constructed as $t_{n+1} = t_n + 2\tau$, with $t_0 \geq \tau$ an arbitrary initial time. Notice that $b(t) \in PC([0,\infty),\mathbb{R})$ and it is smooth almost everywhere. However, its derivative is unbounded. Hence, for this particular regressor and given the unboundedness of $\dot{b}(t)$, the LFs proposed in e.g. Loría et al. (2019a) and Mazenc, de Queiroz et al. (2009) cannot be used. Furthermore, those LFs will present discontinuities, hence, jumps in their level sets.

Moreover, beyond the common restrictions (R1)-(R3), in Mazenc, de Queiroz et al. (2009) it is assumed that (R4) $A(\cdot)$ is smooth, symmetric, bounded and negative definite. In Maghenem and Loría (2017) it is also required that (R5) $m \geq n$ and (R6) $B(\cdot)$ is s.t. $\lambda_{\min}(B^\top(t)B(t))$ is PE, i.e., the regressor $B(t)$ has to be injective "periodically". Otherwise, the function $\psi(t)$ in Maghenem and Loría (2017, Eq. (14)) cannot be PE. In Aranovskiy et al. (2019), a mechanical system is studied and put in the form of (2). In this case, the regressor corresponds to the inertia matrix, which is smooth, (R7) square and of full rank.

Remark 6. In most of the works mentioned above, a nonlinear system instead of a LTV system is considered. In a similar manner, it is possible to use our proposed Lyapunov functions for (1) and (2) to study nonlinear algorithms. The advantage is that, for the nonlinear regressor depending on the states and/or inputs $B(t,\ z(t))$, less restrictive conditions can be obtained.

Additionally, in the literature one can find methods to strictify weak LFs such as Mazenc (2003), Mazenc and Nesic (2007) and Mazenc, Malisoff, and Bernard (2009), summarized in Malisoff and Mazenc (2009). Among these methods, the ones related are described in Malisoff and Mazenc (2009, Chap. 6.1,Thm. 6.1). The basic assumption of Malisoff and Mazenc (2009, Thm. 6.1) is the availability of a weak LF $V(x,t)$ s.t. $\dot{V}(t) \leq -p(t)W(x)$, where $p:\mathbb{R}\to\mathbb{R}_{\geq 0}$ is PE, and $W(x)$ is a positive definite function of x. From (11), one can recognize $p(t)$ as $\lambda_{\min}(C^\top(t)C(t))$ and $W(x)$ as $\|x\|^2$. However, if there are more parameters than measurements ($n>m$), $\lambda_{\min}(C^\top(t)C(t)) = 0\,\forall\, t \geq t_0$, and therefore $p(t)$ is not PE. In the case of (12), and since z_2 does not appear in $\dot{V}(t)$, there is no possible candidate W. Therefore, Malisoff and Mazenc (2009, Thm. 6.1) cannot be applied to any of the systems under study.

5. Proofs of the main results

5.1. Proof of Theorem 1

Equivalence of items (i) and (ii) of Theorem 1 is proved in Anderson (1977, Thm. 1) and Morgan and Narendra (1977b, Thm. 1). Since $0 \leq \sigma - t + T \leq T$ and $C^\top(\sigma)C(\sigma) \geq 0$ in the integration interval $\sigma \in [t - T,\ t]$

$$T\,\gamma_1 \mathbf{I}_n \geq \int_{t-T}^{t} (\sigma - t + T) C^\top(\sigma) C(\sigma) \mathrm{d}\sigma \geq 0\,. \tag{13}$$

$V(x, t)$ can be bounded as

$$\kappa_1 \|x\|^2 \geq V(x,t) \geq \kappa_2 \|x\|^2, \tag{14}$$

$$\begin{aligned} \kappa_1 &= \left(\frac{2(T\, r_1 r_2 \gamma_1)^2}{\gamma_2} + T \right) \lambda_{\max}(\Gamma^{-1}) + T\,\gamma_1, \\ \kappa_2 &= \left(\frac{2(T\, r_1 r_2 \gamma_1)^2}{\gamma_2} + T \right) \lambda_{\min}(\Gamma^{-1}), \end{aligned} \tag{15}$$

so that it is a valid Lyapunov function candidate.

The derivative of $V(x, t)$ along the trajectories of (1) results in

$$\begin{aligned} \dot{V}(x,t) = -x^\top \Bigg(&\frac{2(T\, r_1 r_2 \gamma_1)^2}{\gamma_2} C^\top(t) C(t) \\ &+ 2 \int_{t-T}^{t} (\sigma - t + T) C^\top(\sigma) C(\sigma) \mathrm{d}\sigma\, \Gamma\, C^\top(t) C(t) \\ &+ \int_{t-T}^{t} C^\top(\sigma) C(\sigma) \mathrm{d}\sigma \Bigg) x\,. \end{aligned}$$

The second term in $\dot{V}(x, t)$ can be bounded as

$$\begin{aligned} &\left| x^\top \int_{t-T}^{t} (\sigma - t + T) C^\top(\sigma) C(\sigma) \mathrm{d}\sigma\, \Gamma\, C^\top(t) C(t) x \right| \leq \\ &r_1\, r_2 \left\| \int_{t-T}^{t} (\sigma - t + T) C^\top(\sigma) C(\sigma) \mathrm{d}\sigma \right\| \|C(t)x\|\, \|x\| \leq \\ &T\, r_1\, r_2\, \gamma_1 \|x\|\, \|C(t)x\|\,, \end{aligned}$$

where we have used (13). From the Persistency of Excitation condition (3), i.e., item (ii), we obtain

$$-x^\top \int_{t-T}^{t} C^\top(\sigma) C(\sigma) \mathrm{d}\sigma\, x \leq -\gamma_2 \|x\|^2,$$

so that $\dot{V}(x,t)$ satisfies

$$\dot{V}(x,t) \leq -\frac{2(T\, r_1 r_2 \gamma_1)^2}{\gamma_2} \|C(t)x\|^2 - \gamma_2 \|x\|^2 + 2\, T\, r_1\, r_2\, \gamma_1 \|x\|\, \|C(t)x\|\,.$$

Using Young's inequality for the last term we get

$$\frac{2(T\, r_1 r_2 \gamma_1)^2}{\gamma_2} \|C(t)x\|^2 + \frac{\gamma_2}{2} \|x\|^2 \geq 2\, T\, r_1\, r_2\, \gamma_1 \|x\|\, \|C(t)x\|\,,$$

and then $\dot{V}(x,t) \leq -\gamma_2 \|x\|^2/2 < 0$. This shows that (ii) implies (iii). Now, using Lyapunov's theorem, we conclude that $x = 0$ is GUAS, i.e., (iii) implies (i). □

5.2. Proof of Theorem 3

Equivalence of items (i) and (ii) of Theorem 3 is proved in Anderson (1977, Thm. 2) and Morgan and Narendra (1977a, Thm. 2). The upper and lower bounds of $P(t)$ provided by Lemma 2 implies that $\Pi_1(t)$ in (8) is bounded from above and below by constant positive definite matrices, i.e., $\Pi_1(t)$ is uniformly positive definite. On the other hand and by using similar arguments as those used for obtaining (13) we get the following bounds

$$\begin{aligned} &\left\| \mathcal{P}_{12}(t) \right\| \leq T \int_{t-T}^{t} \int_{s}^{t} \|B(\sigma)\| \mathrm{d}\sigma\, \mathrm{d}s \leq \frac{r_3 T^3}{2}\,, \\ &T\,\gamma_3 \mathbf{I}_n \geq \mathcal{P}_{22}(t) \geq 0\,. \end{aligned}$$

The previous inequalities imply that $\Pi_2(t)$ is bounded. Therefore, for $k > 0$ large enough, it is possible to ensure that $\mathcal{P}(t)$ in (8) is bounded for above and below by constant positive definite matrices, making it uniformly positive definite. Since $V(z, t)$ in (8) is a quadratic form, there exist constants $\bar{\kappa}_1 \geq \bar{\kappa}_2 > 0$ s.t. $\bar{\kappa}_1 \|z\|^2 \geq V(z,t) \geq \bar{\kappa}_2 \|z\|^2$, making $V(z, t)$ a valid Lyapunov function candidate. Now, in order to obtain $\dot{V}(t)$, we need $\dot{\mathcal{P}}_{12}(t)$ and $\dot{\mathcal{P}}_{22}(t)$, which are computed using the Leibniz's rule for differentiation and correspond to

$$\begin{aligned} \dot{\mathcal{P}}_{12}(t) &= -T \cancelto{0}{\mathcal{K}(t,t)} - \int_{t-T}^{t} \frac{\mathrm{d}}{\mathrm{d}t} \left[(s - t + T) \mathcal{K}(t,s) \right] \mathrm{d}s \\ &= \int_{t-T}^{t} \mathcal{K}(t,s) \mathrm{d}s - \int_{t-T}^{t} (s - t + T) B(t) \mathrm{d}s \\ &= \int_{t-T}^{t} \mathcal{K}(t,s) \mathrm{d}s - \frac{T^2}{2} B(t), \end{aligned}$$

$$\begin{aligned} \dot{\mathcal{P}}_{22}(t) &= T \cancelto{0}{\mathcal{K}^\top(t,t) \mathcal{K}(t,t)} \\ &\quad + \int_{t-T}^{t} \frac{\mathrm{d}}{\mathrm{d}t} \left[(s - t + T) \mathcal{K}^\top(t,s) \mathcal{K}(t,s) \right] \mathrm{d}s \\ &= -\int_{t-T}^{t} \mathcal{K}^\top(t,s) \mathcal{K}(t,s) \mathrm{d}s \\ &\quad + \int_{t-T}^{t} (s - t + T) \left(B^\top(t) \mathcal{K}(t,s) + \mathcal{K}^\top(t,s) B(t) \right) \mathrm{d}s \\ &= -\int_{t-T}^{t} \mathcal{K}^\top(t,s) \mathcal{K}(t,s) \mathrm{d}s - B^\top(t) \mathcal{P}_{12}(t) - \mathcal{P}_{12}^\top(t) B(t). \end{aligned}$$

Let $\mathcal{A}(t)$ be as in (2). Then, $\dot{V}(z,t)$ results in $\dot{V}(z,t) = -z^\top \mathcal{Q}(t) z$, with $\mathcal{Q}(t) = -\mathcal{P}(t)\mathcal{A}(t) - \mathcal{A}^\top(t)\mathcal{P}(t) - \dot{\mathcal{P}}(t)$. Define $\mathcal{Q}(t)$ by blocks as

$$\begin{aligned} \mathcal{Q}(t) &= \begin{bmatrix} k\,Q(t) + \mathcal{Q}_{11}(t) & \mathcal{Q}_{12}(t) \\ \mathcal{Q}_{12}^\top(t) & \mathcal{Q}_{22}(t) \end{bmatrix}, \\ \mathcal{Q}_{11}(t) &= \mathcal{P}_{12}(t) B^\top(t) P(t) + P(t) B(t) \mathcal{P}_{12}^\top(t)\,, \\ \mathcal{Q}_{12}(t) &= -A^\top(t) \mathcal{P}_{12}(t) + P(t) B(t) \mathcal{P}_{22}(t) + \frac{T^2}{2} B(t) \\ &\quad - \int_{t-T}^{t} \mathcal{K}(t,s) \mathrm{d}s\,, \\ \mathcal{Q}_{22}(t) &= \int_{t-T}^{t} \mathcal{K}^\top(t,s) \mathcal{K}(t,s) \mathrm{d}s\,. \end{aligned}$$

Recall that all these matrices are bounded. Due to the *persistency excitation* condition (7), i.e., item (ii) of Theorem 3, $\mathcal{Q}_{22}(t)$ is positive definite. Thus, using the Schur complement, $\mathcal{Q}(t)$ is positive definite if the matrix $k\,Q(t) + \mathcal{Q}_{11}(t) - \mathcal{Q}_{12}(t)\,\mathcal{Q}_{22}^{-1}(t)\,\mathcal{Q}_{12}^\top(t)$ is positive definite. This will be the case if

$$\begin{aligned} &\zeta^\top \left(k\,Q(t) + \mathcal{Q}_{11}(t) - \mathcal{Q}_{12}(t)\,\mathcal{Q}_{22}^{-1}(t)\,\mathcal{Q}_{12}^\top(t) \right) \zeta \geq \\ &\quad \left(k\,\gamma_7 - \|\mathcal{Q}_{11}(t)\| - \frac{1}{\gamma_4} \|\mathcal{Q}_{12}(t)\|^2 \right) \|\zeta\|^2 \geq \epsilon^2 \|\zeta\|^2 \end{aligned}$$

is valid for some $\epsilon \neq 0$. Since $\mathcal{Q}_{11}(t)$ and $\mathcal{Q}_{12}(t)$ are bounded and independent of k, the latter inequality is satisfied for $k > 0$ sufficiently large. Therefore, $\dot{V}(z,t)$ is negative definite. This shows that (ii) implies (iii). From Lyapunov's theorem we conclude that $z = 0$ is GUAS, i.e., (iii) implies (i). □

6. Conclusions

Strong Lyapunov functions that work under necessary and sufficient conditions that ensure GUAS of two classical systems in adaptive control are presented for the first time. It is the hope of the authors that the availability of these functions allows to analyze the effect of noise, parameter variations and nonlinearities in adaptive control systems, and that they help in the design of tuning rules to obtain specific convergence rates.

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