International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: https://www.tandfonline.com/loi/tcon20

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INTERNATIONAL JOURNAL OF CONTROL, 2016
VOL. 89, NO. 5, 909–918
http://dx.doi.org/10.1080/00207179.2015.1104555

Robustness of delayed multistable systems with application to droop-controlled inverter-based microgrids

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ABSTRACT
Motivated by the problem of phase-locking in droop-controlled inverter-based microgrids with delays, the recently developed theory of input-to-state stability (ISS) for multistable systems is extended to the case of multistable systems with delayed dynamics. Sufficient conditions for ISS of delayed systems are presented using Lyapunov–Razumikhin functions. It is shown that ISS multistable systems are robust with respect to delays in a feedback. The derived theory is applied to two examples. First, the ISS property is established for the model of a nonlinear pendulum and delay-dependent robustness conditions are derived. Second, it is shown that, under certain assumptions, the problem of phase-locking analysis in droop-controlled inverter-based microgrids with delays can be reduced to the stability investigation of the nonlinear pendulum. For this case, corresponding delay-dependent conditions for asymptotic phase-locking are given.

ARTICLE HISTORY
Received 8 June 2015
Accepted 3 October 2015

KEYWORDS
Input-to-state stability; delay nonlinear systems; microgrids

1. Introduction

The increasing penetration of renewable distributed generation (DG) units at the low and medium voltage levels has a strong impact on the power system structure (Fang, Misra, Xue, & Yang, 2012; Farhangi, 2010; Varaiya, Wu, & Bialek, 2011). This fact requires new control and operation strategies to ensure a reliable and efficient electrical power supply (Farhangi, 2010; Green & Prodanovic, 2007). An emerging concept to address these challenges is the microgrid (Farhangi, 2010; Hatziargyriou, Asano, Iravani, & Marnay, 2007; Lasseter, 2002). A microgrid is a locally controllable subset of a larger electrical network. It is composed of several DG units, storage devices and loads.

Typically, most DG units in an AC microgrid are connected to the network via AC inverters (Green & Prodanovic, 2007). Under ideal conditions, an inverter-based DG unit can be modelled as an ideal controllable voltage source (Lopes, Moreira, & Madureira, 2006; Rocabert, Luna, Blaabjerg, & Rodriguez, 2012). Furthermore, a popular control scheme to operate inverter-based DG units with the purpose to achieve frequency synchronisation and power sharing in the network is droop control (Chandorkar, Divan, & Adapa, 1993; Guerrero, Loh, Chandorkar, & Lee, 2013). Conditions for stability in droop-controlled microgrids with inverters modelled as ideal controllable voltage sources have been derived, e.g. in Simpson-Porco, Dörfler, and Bullo (2013); Schiffer, Ortega, Astolfi, Raisch, and Sezi (in press); Münz and Romeres (2013).

In general, inverter-based microgrids operated with droop control have several equilibria (Simpson-Porco et al., 2013; Schiffer et al., 2014). Thus, they are multistable systems. Stability analysis (Angeli, Ferrell, & Sontag, 2004; Efimov & Fradkov, 2009; Monzón & Potrie, 2006; Nitecki & Shub, 1975; Rajaram, Vaidya, & Fardad, 2007; Rantzer, 2001; Rumyantsev & Oziraner, 1987; Smale, 1967; van Handel, 2006) and robust stability analysis (Angeli, 2004; Angeli & Efimov, 2015; Angeli & Praly, 2011; Chaves, Eissing, & Allgower, 2008; Stan & Sepulchre, 2007) for this class of systems is rather complicated. Recently, the input-to-state stability (ISS) theory (Sontag, 1995) has been extended to multistable systems in Angeli and Efimov (2013), Angeli and Efimov (2015) (see also Kellett, Wirth, and Dower (2013) for discussion on ISS property with respect to an unbounded set).

Furthermore, in a practical setup, the droop control scheme is applied to an inverter by means of digital discrete time control. Besides clock drifts, see, e.g. Schiffer, Ortega, Hans, and Raisch (2015), digital control usually introduces time delays (Kukrer, 1996; Maksimovic & Zane, 2007; Nussbaumer, Heldwein, Gong, Round, & Kolar, 2008). According to Nussbaumer et al. (2008), the

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main reasons for this are (1) sampling of control variables, (2) calculation time of the digital controller and (3) generation of the pulse-width modulation. We refer the reader to, e.g. Nussbaumer et al. (2008) for further details. To the best of the authors' knowledge this fact has yet not been considered in previous analysis of droop-controlled microgrids.

Motivated by the aforementioned phenomenon, the main theoretical contribution of the present paper is to extend the recently derived ISS framework for multistable systems (Angeli & Efimov, 2013, 2015) to multistable systems with delay (some preliminary consideration has been performed by Efimov, Ortega, and Schiffer (2015)). In particular, sufficient conditions for ISS of multistable systems in the presence of delays are given in terms of a Lyapunov–Razumikhin function. It is also shown that ISS multistable systems are robust with respect to feedback delays. This result is illustrated via the example of a nonlinear pendulum. We would like to point out that related works on ISS of time-delay systems by employing Lyapunov functions (Dashkovskiy & Naujok, 2010; Fridman, 2014; Pepe, Karafyllis, & Jiang, 2008) is limited to systems with a single equilibrium point or a compact attracting set. Next, based on the established results, we provide the main practical contribution: a condition for asymptotic phase-locking in a microgrid composed of two droop-controlled inverters with delay. The analysis is conducted for a simplified inverter model derived under the assumptions of constant voltage amplitudes and ideal clocks, as well as negligible dynamics of the internal inverter filter and controllers. In that scenario, the delay merely affects the phase angle of the inverter output voltage. The stability results are illustrated by simulations.

2. Preliminaries

For an n-dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold M without a boundary, let the map $f(x, d) : M \times \mathbb{R}^m \to T_xM$ be of class $\mathcal{C}^1$, and consider a nonlinear system of the following form:

$$\dot{x}(t) = f(x(t), d(t)), \qquad (1)$$

where the state $x \in M$ and $d(t) \in \mathbb{R}^m$ (the input $d(\cdot)$ is a locally essentially bounded and measurable signal) for $t \geq 0$. We denote by $X(t, x_0; d)$ the uniquely defined solution of (1) at time t fulfilling $X(0, x_0; d) = x_0$. Together with (1) we will analyse its unperturbed version:

$$\dot{x}(t) = f(x(t), 0). \qquad (2)$$

A set $S \subset M$ is invariant for the unperturbed system (2) if $X(t, x; 0) \in S$ for all $t \in \mathbb{R}$ and for all $x \in S$. Define the distance from a point $x \in M$ to the set $S \subset M$ as $|x|_S = \min_{a \in S} \delta(x, a)$, where the symbol $\delta(x_1, x_2)$ denotes the Riemannian distance between x_1 and x_2 in M, $|x| = |x|_{\{0\}}$ for $x \in M$ or a usual euclidean norm of a vector $x \in \mathbb{R}^n$. For a signal $d : \mathbb{R} \to \mathbb{R}^m$ the essential supremum norm is defined as $\|d\|_\infty = \text{ess sup}_{t \geq 0} |d(t)|$.

2.1 Decomposable sets

Let $\Lambda \subset M$ be a compact invariant set for (2).

Definition 2.1 (Nitecki & Shub, 1975)**:** A decomposition of Λ is a finite and disjoint family of compact invariant sets $\Lambda_1, \ldots, \Lambda_k$ such that

$$\Lambda = \bigcup_{i=1}^{k} \Lambda_i.$$

For an invariant set Λ, its attracting and repulsing subsets are defined as follows:

$$\mathfrak{A}(\Lambda) = \{x \in M : |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to +\infty\},$$
$$\mathfrak{R}(\Lambda) = \{x \in M : |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to -\infty\}.$$

Define a relation on $\mathcal{W} \subset M$ and $\mathcal{D} \subset M$ by $\mathcal{W} \prec \mathcal{D}$ if $\mathfrak{A}(\mathcal{W}) \cap \mathfrak{R}(\mathcal{D}) \neq \emptyset$.

Definition 2.2 (Nitecki & Shub, 1975)**:** Let $\Lambda_1, \ldots, \Lambda_k$ be a decomposition of Λ, then

1. An r-cycle ($r \geq 2$) is an ordered r-tuple of distinct indices $i_1, \ldots, i_r$ such that $\Lambda_{i_1} \prec \ldots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.
2. A 1-cycle is an index i such that $[\mathfrak{R}(\Lambda_i) \cap \mathfrak{A}(\Lambda_i)] - \Lambda_i \neq \emptyset$.
3. A filtration ordering is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \Rightarrow i \leq j$.

As we can conclude from Definition 2.2, existence of an r-cycle with $r \geq 2$ is equivalent to existence of a heteroclinic cycle for (2) (Guckenheimer & Holmes, 1988). Furthermore, existence of a 1-cycle implies existence of a homoclinic cycle for (2) (Guckenheimer & Holmes, 1988).

Definition 2.3: The set $\mathcal{W}$ is called decomposable if it admits a finite decomposition without cycles, $\mathcal{W} = \bigcup_{i=1}^{k} \mathcal{W}_i$, for some non-empty disjoint compact sets $\mathcal{W}_i$, which form a filtration ordering of $\mathcal{W}$, as detailed in Definitions 2.1 and 2.2.

2.2 Robustness notions

The following robustness notions for systems represented by (1) have been introduced in Angeli and Efimov (2013,

2015) (see also Dashkovskiy, Efimov, and Sontag (2011) for a survey on ISS framework).

Definition 2.4: We say that the system (1) has the practical asymptotic gain (pAG) property if there exist $\eta \in \mathcal{K}_\infty$ and a non-negative real q such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\limsup_{t\to+\infty} |X(t,x;d)|_{\mathcal{W}} \leq \eta(\|d\|_\infty) + q.$$

If $q = 0$, then we say that the asymptotic gain (AG) property holds.

Definition 2.5: We say that the system (1) has the limit property (LIM) with respect to $\mathcal{W}$ if there exists $\mu \in \mathcal{K}_\infty$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\inf_{t\geq 0} |X(t,x;d)|_{\mathcal{W}} \leq \mu(\|d\|_\infty).$$

Definition 2.6: We say that the system (1) has the practical global stability (pGS) property with respect to $\mathcal{W}$ if there exist $\beta \in \mathcal{K}_\infty$ and $q \geq 0$ such that for all $x \in M$ and measurable essentially bounded inputs $d(\cdot)$ the following holds for all $t \geq 0$:

$$|X(t,x;d)|_{\mathcal{W}} \leq q + \beta(\max\{|x|_{\mathcal{W}}, \|d\|_\infty\}).$$

It has been shown in Angeli and Efimov (2013, 2015) that to characterise pAG property in terms of Lyapunov functions the following notion is appropriate.

Definition 2.7: We say that a $\mathcal{C}^1$ function $V: M \to \mathbb{R}$ is a practical ISS–Lyapunov function for (1) if there exist $\mathcal{K}_\infty$ functions α_1, $[\alpha_2]$, α_3 and γ, and scalar $q \geq 0$ [and $c \geq 0$] such that

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq [\alpha_2(|x|_{\mathcal{W}} + c)],$$

the function V is constant on each $\mathcal{W}_i$ and the following dissipation holds:

$$DV(x)f(x,d) \leq -\alpha_3(|x|_{\mathcal{W}}) + \gamma(|d|) + q.$$

If the latter inequality holds for $q = 0$, then V is said to be an ISS–Lyapunov function.

Notice that α_2 and c are within square brackets as their existence follows (without any additional assumptions) by standard continuity arguments.

The main result of Angeli and Efimov (2013, 2015) connecting these robust stability properties is stated below; it extends the results of Sontag and Wang (1995, 1996) obtained for connected sets.

Theorem 2.1: *Consider a nonlinear system as in (1) and let a compact invariant set containing all α- and ω-limit sets of (2) $\mathcal{W}$ be decomposable (in the sense of Definition 2.3). Then the following facts are equivalent.*

1. *The system admits an ISS Lyapunov function;*
2. *The system enjoys the AG property;*
3. *The system admits a practical ISS Lyapunov function;*
4. *The system enjoys the pAG property;*
5. *The system enjoys the LIM property and the pGS.*

Definition 2.8 (Angeli & Efimov, 2015): Suppose that a nonlinear system as in (1) satisfies the assumptions and the list of equivalent properties of Theorem 2.1. Then this system is called ISS with respect to the set $\mathcal{W}$.

3. Multistable systems with delays

Let $\tau > 0$, for a function $d: [-\tau, +\infty) \to \mathbb{R}^m$ and $t \geq 0$ denote a function $d_t(\cdot): [-\tau, 0] \to \mathbb{R}^m$ defined by $d_t(\theta) = d(t+\theta)$ for $\theta \in [-\tau, 0]$. Denote by $\mathcal{D}$ a set of bounded and piecewise continuous functions $d_t(\cdot): [-\tau, 0] \to \mathbb{R}^m$. Consider a functional differential equation on an n-dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold M without a boundary:

$$\dot{x}(t) = F(x_t, d_t), \quad x_0 \in \mathcal{C}_\tau, \tag{3}$$

where the map $F: \mathcal{C}_\tau \times \mathcal{D} \to T_xM$ is of class $\mathcal{C}^1$ (we will denote a set of continuous functions $\xi: [-\tau, 0] \to M$ by $\mathcal{C}_\tau$), $x(t) \in M$ is the state, $x_t \in \mathcal{C}_\tau$ and $d_t \in \mathcal{D}$ for all $t \geq 0$. We denote by $X(t, x_0; d)$ the uniquely defined solution of (3) at time t fulfilling $X(\theta, x_0; d) = x_0(\theta)$ for all $\theta \in [-\tau, 0]$; $X_t^{x_0,d}(\theta) = X(t+\theta, x_0; d)$ for $\theta \in [-\tau, 0]$. Define as in Teel (1998)

$$|x_t| = \max_{\theta\in[-\tau,0]} |x(t+\theta)|, \quad ||x||_{t_0} = \sup_{t\geq t_0} |x_t| = \sup_{t\geq t_0-\tau} |x(t)|.$$

Again, together with (3), we will analyse its unperturbed version:

$$\dot{x}(t) = F(x_t, 0). \tag{4}$$

A set $\mathcal{S} \subset \mathcal{C}_\tau$ is invariant for the unperturbed system (4) if $X_t^{x_0,0} \in \mathcal{S}$ for all $t \in \mathbb{R}_+$ and for all $x_0 \in \mathcal{S}$. Define the distance from a function $\xi \in \mathcal{C}_\tau$ to a set $\mathcal{S} \subset \mathcal{C}_\tau$ as $||\xi||_{\mathcal{S}} = \min_{\alpha\in\mathcal{S}} |\xi - \alpha|$.

Let $\mathcal{W} \subset M$ be a set, denote by $\overbrace{\mathcal{W}}$ a subset of $\overline{\mathcal{W}} = \{\xi \in \mathcal{C}_\tau : \xi(t) \in \mathcal{W}\ \forall t \in [-\tau, 0]\}$ such that if $\zeta \in \overbrace{\mathcal{W}}$ then $\zeta = X_\tau^{\xi,0}$ for $\xi \in \overline{\mathcal{W}}$. For stability analysis in time-delay systems, it is required to define a distance to invariant sets in two spaces: in $\mathbb{R}^n$ with respect to the set $\mathcal{W}$ and in $\mathcal{C}_\tau$ with respect to corresponding invariant set $\overbrace{\mathcal{W}}$

(functions from $\mathcal{C}_\tau$ taking values in $\mathcal{W}$ and solutions of (3)). The following stability notions for (3) are considered in this work (for a recent survey on stability tools for time-delay systems, see Fridman (2014)).

Definition 3.1: The system (3) has the pAG property with respect to the set $\mathcal{W}$ if there exist $\eta \in \mathcal{K}_\infty$ and a non-negative real q such that for all $x_0 \in \mathcal{C}_\tau$ and all bounded piecewise continuous inputs $d(\cdot)$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\limsup_{t\to+\infty} |X(t, x_0; d)|_{\mathcal{W}} \leq \eta(\|d_t\|_0) + q.$$

If $q = 0$, then we say that the AG property holds.

This property can be equivalently stated as

$$\limsup_{t\to+\infty} ||X_t^{x_0,d}||_{\underbrace{}_{\mathcal{W}}} \leq \eta(\|d_t\|_0) + q$$

and it implies that (a subset of) $\overbrace{\mathcal{W}}$ is invariant for (4) if $q = 0$.

Definition 3.2: The system (3) has the pGS property with respect to the set $\mathcal{W}$ if there exist $\beta \in \mathcal{K}_\infty$ and $q \geq 0$ such that for all $x_0 \in \mathcal{C}_\tau$ and all bounded piecewise continuous inputs $d(\cdot)$ the following holds for all $t \geq 0$:

$$|X(t, x_0; d)|_{\mathcal{W}} \leq q + \beta(\max\{||x_0||_{\underbrace{}_{\mathcal{W}}}, \|d_t\|_0\}).$$

To characterise pAG and pGS properties for a time-delay system (3) the Lyapunov–Razumikhin approach is used in this work (Dashkovskiy & Naujok, 2010; Pepe et al., 2008). Given a continuous function $x: [-\tau, +\infty) \to M$ with a $\mathcal{C}^1$ function $U: M \to \mathbb{R}$ denote $U(t) = U(x(t))$, if $x(t) = X(t, x_0; d)$ is a solution to (3) for some piecewise continuous $d: [-\tau, +\infty) \to \mathbb{R}^m$ and initial condition $x_0 \in \mathcal{C}_\tau$, then the upper right-hand side derivative of U along this solution is

$$D^+U(t) = \limsup_{h\to 0^+} \frac{U(t+h) - U(t)}{h}.$$

Definition 3.3: A $\mathcal{C}^1$ function $U: M \to \mathbb{R}$ is a practical ISS–Lyapunov–Razumikhin (ISS–LR) function for (3) if there exist $\mathcal{K}_\infty$ functions α_1, $[\alpha_2]$, α_4, γ and γ_U, $\gamma_U(s) < s$ for all $s > 0$, and scalar $q \geq 0$ [and $c \geq 0$] such that

$$\begin{aligned} &\alpha_1(|x|_{\mathcal{W}}) \leq U(x) \leq [\alpha_2(|x|_{\mathcal{W}} + c)], \\ &U(t) \geq \max\{\gamma_U(|U_t|), \gamma(|d_t|), q\} \Rightarrow \\ &\qquad D^+U(t) \leq -\alpha_4[U(t)]. \end{aligned}$$

If the latter inequality holds for $q = 0$, then U is said to be an ISS–LR function.

Definition 3.4: The system in (3) is said to be ISS with respect to the set $\mathcal{W}$ if it admits pAG and pGS properties with respect to the set $\mathcal{W}$.

Note that Definitions 2.8 and 3.4 introduce the same property, but for different classes of systems, (1) and (3), respectively. The following result can be stated connecting pAG, pGS properties and the existence of an ISS–LR function.

Theorem 3.1: *Consider the system (3). Suppose there exists an ISS–LR function $U: M \to \mathbb{R}$ as in Definition 3.3. Then the system (3) admits the pAG property from Definition 3.1 with $\eta(s) = \alpha_1^{-1} \circ \gamma(s)$ and the pGS property from Definition 3.2.*

Proof. By using Lemma 1 of Teel (1998) and Definition 3.3, it follows that for all $t \geq 0$:

$$U(t) \leq \max\{\chi[U(0), t], \gamma_U(||U_t||_0), \gamma(||d_t||_0), q\} \tag{5}$$

for some $\chi \in \mathcal{KL}$ ($\chi(s, 0) \geq s$ for all $s \geq 0$). In addition, for $\varphi(s) = 0.5[1 - \text{sign}(s - \tau)]$:

$$|U_t| \leq \max\{|U_0|\varphi(t), \sup_{t\geq 0} U(t)\} \tag{6}$$

for all $t \geq 0$ (this estimate is true by definition of the norm). By substituting (5) in (6) and taking the supremum of the left-hand side of (6) we obtain:

$$\begin{aligned} ||U_t||_0 \leq \max\{&|U_0|\varphi(t), \chi[U(0), 0], \gamma_U(||U_t||_0), \\ &\gamma(||d_t||_0), q\} \\ \leq \max\{&\chi(|U_0|, 0), \gamma_U(||U_t||_0), \gamma(||d_t||_0), q\}. \end{aligned}$$

Since $\gamma_U(s) < s$ for all $s > 0$ by assumption, the boundedness of $U(t)$ can be proven (Teel, 1998), i.e.

$$||U_t||_0 \leq \max\{\chi(|U_0|, 0), \gamma(||d_t||_0), q\}.$$

From Definition 3.3, this estimate implies (recall that $\alpha(a + b) \leq \alpha(2a) + \alpha(2b)$ for a function $\alpha \in \mathcal{K}$ for any $a \geq 0$, $b \geq 0$)

$$\begin{aligned} |X(t, x_0; d)|_{\mathcal{W}} &\leq \alpha_1^{-1}[\max\{\chi(\alpha_2(||x_0||_{\underbrace{}_{\mathcal{W}}} + c), 0), \\ &\qquad \gamma(||d_t||_0), q\}] \\ &\leq \max\{\alpha_1^{-1} \circ \chi(\alpha_2(2||x_0||_{\underbrace{}_{\mathcal{W}}}), 0), \\ &\qquad \alpha_1^{-1} \circ \gamma(||d_t||_0)\} \\ &\quad + \alpha_1^{-1} \circ \chi(\alpha_2(2c), 0) + \alpha_1^{-1}(q) \\ &\leq \beta(\max\{||x_0||_{\underbrace{}_{\mathcal{W}}}, ||d_t||_0\}) + \tilde{q}, \end{aligned}$$

where $\beta(s) = \alpha_1^{-1} \circ \max\{\chi[\alpha_2(2s), 0], \gamma(s)\}$ and $\tilde{q} = \alpha_1^{-1} \circ \chi(\alpha_2(2c), 0) + \alpha_1^{-1}(q)$. This implies that the system (3) possesses the pGS property.

To prove asymptotic convergence consider $t > 2\tau$, then by substituting (6) in (5) we obtain:

$$U(t) \leq \max\{\chi[U(0), t], \gamma_U[\sup_{t\geq 0} U(t)], \gamma(||d_t||_0), q\}.$$

Again $\gamma_U(s) < s$ for all $s > 0$, and for a sufficiently big $T > 0$ (Teel, 1998):

$$\sup_{t\geq 2\tau+T} U(t) \leq \max\{\chi[U(0), t], \gamma(||d_t||_0), q\}.$$

Next, by repeating the arguments given above, asymptotic convergence can be proven, *i.e.*,

$$\limsup_{t\to+\infty} U(t) \leq \max\{\gamma(\|d_t\|_0), q\}, \tag{7}$$

or, equivalently,

$$\limsup_{t\to+\infty} |X(t, x_0; d)|_{\mathcal{W}} \leq \alpha_1^{-1} \circ \gamma(\|d_t\|_0)] + \alpha_1^{-1}(q)$$

completing the proof. □

4. ISS of multistable systems with delayed perturbations

In this section we consider the robustness of the system (1) with respect to a disturbance d, which is dependent on a delayed state. The analysis is conducted under the assumption that the system (1) is ISS with respect to a set $\mathcal{W}$. Furthermore, the proposed approach is illustrated via the well-known example of a nonlinear pendulum with delay.

4.1 Robustness analysis

If (1) is ISS with respect to the set $\mathcal{W}$, then by Theorem 2.1 there exists an ISS Lyapunov function V as in Definition 2.7. From the inequalities $\alpha_3[0.5\alpha_2^{-1} \circ V(x)] \leq \alpha_3(0.5[|x|_{\mathcal{W}} + c]) \leq \alpha_3(|x|_{\mathcal{W}}) + \alpha_3(c)$ we obtain

$$DV(x)f(x, d) \leq -\alpha_4[V(x)] + \gamma(|d|) + \tilde{q},$$

where $\alpha_4(s) = \alpha_3[0.5\alpha_2^{-1}(s)]$ and $\tilde{q} = q + \alpha_3(c)$.

Assume that the input d has two terms d_1 and d_2, and d_2 is a function of $x_t \in \mathcal{C}_\tau$ for some $\tau > 0$, i.e.:

$$d = d_1 + d_2,\ \ d_2 = g(x_t), \tag{8}$$

where g is a continuous function, $|g(x_t)| \leq \upsilon(|V_t|) + \upsilon_0$ for $\upsilon \in \mathcal{K}_\infty$ and $\upsilon_0 \geq 0$. Denote further for simplicity of notation $d = d_1$, then the system (1) is transformed to (3) with

$$F(x_t, d_t) = f(x(t), d + g(x_t)),$$

and

$$D^+V(t) \leq -\alpha_4(V(t)) + \gamma(2\upsilon(|V_t|) + 2\upsilon_0) + \gamma(2|d_t|) + \tilde{q}.$$

This estimate can be rewritten as follows:

$$\begin{gathered} V(t) \geq \max\{\hat{\gamma}_V(|V_t|), \hat{\gamma}(|d_t|), \hat{q}\} \Rightarrow \\ D^+V(t) \leq -0.5\alpha_4(V(t)), \\ \hat{\gamma}_V(s) = \alpha_4^{-1}[6\gamma(4\upsilon(s))],\ \ \hat{\gamma}(s) = \alpha_4^{-1}[6\gamma(2s)], \\ \hat{q} = \alpha_4^{-1}[6\tilde{q} + 6\gamma(4\upsilon_0)]. \end{gathered}$$

It is straightforward to see that if $\hat{\gamma}_V(s) < s$ for all $s > 0$, then V is an ISS–LR function for (1) with (8), and by Theorem 3.1 this system possesses pAG and pGS properties.

4.2 Illustration for a nonlinear pendulum

Now, the procedure for a robust ISS analysis of a multistable system with delays outlined in Section 4.1 is illustrated via the example of a nonlinear pendulum. First, we prove the assumption made in Section 4.1 that the pendulum is ISS with respect to a set $\mathcal{W}$. Second, a condition for ISS of a pendulum with delay is derived. During our analysis, we also establish almost global attractivity of an equilibrium of a nonlinear pendulum with constant nonzero input. To the best of our knowledge, such result is not available in the literature thus far.

4.21 Delay-free case

Consider a nonlinear pendulum:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\Omega^2 \sin(x_1) - \kappa x_2 + d, \end{aligned} \tag{9}$$

where the state $x = [x_1, x_2]$ takes values on the cylinder $M := \mathbb{S} \times \mathbb{R}$, $d(t) \in \mathbb{R}$ is an exogenous disturbance, and Ω, κ are constant positive parameters. The total energy of (9) is $H(x) = 0.5x_2^2 + \Omega^2(1 - \cos(x_1))$ and $\dot{H} = x_2 d - \kappa x_2^2$. The unperturbed system (9) has two equilibria $[0, 0]$ and $[\pi, 0]$ (the former is attractive and the latter one is a saddle-point). Thus, $\mathcal{W} = \{[0, 0] \cup [\pi, 0]\}$ is a compact set containing all α- and ω-limit sets of (9) for $d = 0$. In addition, it is straightforward to check that $\mathcal{W}$ is decomposable in the sense of Definition 2.3.

Lemma 4.1: *The system (9) is ISS with respect to the set $\mathcal{W}$.*

Proof. Following the ideas of Angeli and Praly (2011), consider a Lyapunov function candidate

$$V(x) = H(x) + \kappa\epsilon(1 - \cos(x_1)) + \epsilon x_2 \sin(x_1), \quad (10)$$

which is positive definite for any $0.5[\kappa - \sqrt{\kappa^2 + 4\Omega^2}] \le \epsilon \le 0.5[\kappa + \sqrt{\kappa^2 + 4\Omega^2}]$. Indeed, $1 - \cos(x_1) \ge 0.5(1 - \cos^2(x_1))$ for any $x_1 \in \mathbb{S}$, then

$$\begin{aligned} V(x) &= 0.5x_2^2 + [\Omega^2 + \kappa\epsilon](1 - \cos(x_1)) + \epsilon x_2 \sin(x_1) \\ &\ge 0.5x_2^2 + 0.5[\Omega^2 + \kappa\epsilon](1 - \cos^2(x_1)) \\ &\quad + \epsilon x_2 \sin(x_1) \\ &= 0.5x_2^2 + 0.5[\Omega^2 + \kappa\epsilon]\sin^2(x_1) + \epsilon x_2 \sin(x_1) \\ &= \frac{1}{2}\begin{bmatrix} \sin(x_1) & x_2 \end{bmatrix} Y \begin{bmatrix} \sin(x_1) \\ x_2 \end{bmatrix}, \\ Y &= \begin{bmatrix} \Omega^2 + \kappa\epsilon & \epsilon \\ \epsilon & 1 \end{bmatrix}, \end{aligned}$$

and the matrix Y is positive definite for $2\epsilon \in [\kappa - \sqrt{\kappa^2 + 4\Omega^2}, \kappa + \sqrt{\kappa^2 + 4\Omega^2}]$. Next,

$$\begin{aligned} \dot V &= x_2 d - \kappa x_2^2 + \kappa\epsilon \sin(x_1)x_2 + \epsilon x_2^2 \cos(x_1) \\ &\quad + \epsilon \sin(x_1)[d - \Omega^2 \sin(x_1) - \kappa x_2] \\ &= -[\kappa - \epsilon\cos(x_1)]x_2^2 - \epsilon\Omega^2 \sin^2(x_1) \\ &\quad + \epsilon \sin(x_1)d + x_2 d. \end{aligned} \quad (11)$$

Thus, for further consideration select $0 < \epsilon \le 0.5[\kappa + \sqrt{\kappa^2 + 4\Omega^2}]$, then

$$\begin{aligned} \dot V &\le -0.5[\kappa - \epsilon]x_2^2 - 0.5\epsilon\Omega^2 \sin^2(x_1) \\ &\quad + 0.5\left[\epsilon\Omega^{-2} + \frac{1}{\kappa - \epsilon}\right]d^2 \end{aligned} \quad (12)$$

and V is an ISS–Lyapunov function for (9), provided that $0 < \epsilon < \min\{\kappa, 0.5[\kappa + \sqrt{\kappa^2 + 4\Omega^2}]\} = \kappa$. The result follows from Theorem 2.1. □

Using this result, it is possible to prove that for a constant input d (with $d < \Omega^2$), the pendulum still has two steady-state points with similar stability properties.

Lemma 4.2: *Let* $d < \min\{\Omega^2, \sqrt{\frac{\varrho\lambda_{\min}(Y)}{2}}\frac{\pi}{\zeta}, 0.5\sqrt{\epsilon}\Omega\frac{\pi}{\zeta}, \xi\}$ *be a constant input in* (9), *where*

$$\begin{aligned} \varrho &= \min\left[\frac{\kappa - \epsilon}{1 + \epsilon}, \frac{1}{\sqrt{2\pi}}\frac{\epsilon\Omega^2}{(\Omega^2 + (\kappa + 1)\epsilon)}\right], \\ \zeta &= \sqrt{\sqrt{2\pi}[\epsilon\Omega^{-2} + \frac{1}{\kappa - \epsilon}]}, \\ \xi &= \frac{2\sqrt{\Omega^2 + \kappa\epsilon}}{\zeta}\left(\frac{\Omega^2 + (\kappa + 1)\epsilon}{\epsilon\Omega^2} + \frac{1}{\sqrt{2\pi}\varrho}\right) \end{aligned}$$

and $0 < \epsilon < \min\{1, \kappa\}$ *is a parameter. Then the system has two equilibria,* $[\arcsin(d\Omega^{-2}), 0]$ *and* $[\pi - \arcsin(d\Omega^{-2}), 0]$, *the former one is almost globally attractive.*

Proof. It is straightforward to verify that for $d < \Omega^2$ the system (9) has two equilibrium points in M. In order to show that there is no another invariant solution and to investigate stability of these equilibriums we can use ISS property established in Lemma 4.1. From the proof of Lemma 4.1 we obtain that for an ISS–Lyapunov function V in (10) its derivative satisfies (12) for $0 < \epsilon < \kappa$. Additionally, from (10) and (12) we have the following series of inequalities:

$$\begin{aligned} &V(x) \le 0.5[(1 + \epsilon)x_2^2 + (\Omega^2 + (\kappa + 1)\epsilon)x_1^2] \quad \forall x \in M, \\ &\quad \dot V \le -0.5[\kappa - \epsilon]x_2^2 - 0.5\frac{\epsilon\Omega^2}{\sqrt{2\pi}}x_1^2 \\ &\qquad + 0.5\left[\epsilon\Omega^{-2} + \frac{1}{\kappa - \epsilon}\right]d^2 \quad \forall x \in M_1, \\ &V(x) \ge 0.5[(1 - \epsilon)x_2^2 - (\Omega^2 + (\kappa + 1)\epsilon)(x_1 - \pi)^2] \\ &\qquad + 2(\Omega^2 + \kappa\epsilon) \quad \forall x \in M_2, \\ &\quad \dot V \le -0.5[\kappa - \epsilon]x_2^2 - 0.5\frac{\epsilon\Omega^2}{\sqrt{2\pi}}(x_1 - \pi)^2 \\ &\qquad + 0.5\left[\epsilon\Omega^{-2} + \frac{1}{\kappa - \epsilon}\right]d^2 \quad \forall x \in M_2, \end{aligned}$$

where $M_1 := \{x \in M\colon |x_1| \le 0.5\pi\}$ and $M_2 := \{x \in M\colon 0.5\pi \le x_1 \le 1.5\pi\}$. Then from the first two inequalities we obtain for all $x \in M_1$:

$$\dot V \le -\varrho V + 0.5\left[\epsilon\Omega^{-2} + \frac{1}{\kappa - \epsilon}\right]d^2$$

and $M_{d,0} = \{x \in M : V(x) \le 0.5\varrho^{-1}[\frac{\epsilon}{\Omega^2} + \frac{1}{\kappa-\epsilon}]d^2\}$ is an invariant set if it is contained in M_1. From the proof of Lemma 4.1 we know that

$$V(x) \ge \lambda_{\min}(Y)(x_2^2 + \sin^2(x_1)),$$

where $\lambda_{\min}(Y) = 0.5(\Omega^2 + \kappa\epsilon + 1 - \sqrt{(\Omega^2 + \kappa\epsilon - 1)^2 + 4\epsilon^2})$ is the minimal eigenvalue of the matrix Y defined there. Since $V(x) \ge \lambda_{\min}(Y)\sin^2(x_1) \ge \frac{\lambda_{\min}(Y)}{\sqrt{2\pi}}x_1^2$ for $x \in M_1$, then we obtain that $M_{d,0} \subset M_1$ if $|x_1| \le \sqrt{\frac{\sqrt{2\pi}}{2\lambda_{\min}(Y)}\varrho^{-1}[\frac{\epsilon}{\Omega^2} + \frac{1}{\kappa-\epsilon}]}|d| \le \frac{\pi}{2}$ or $|d| \le \sqrt{\frac{\varrho\lambda_{\min}(Y)}{2}}\frac{\pi}{\zeta}$. In this case $V(x) \le \frac{\pi^{3/2}}{4\sqrt{2}}\lambda_{\min}(Y)$ for $x \in M_1$.

In the set M_2 the function V has a saddle point at $[\pi, 0]$ and the derivative of V is negative-definite outside the set $M_{d,\pi} = \{x \in M_2 : [\kappa - \epsilon]x_2^2 + \frac{\epsilon\Omega^2}{\sqrt{2\pi}}(x_1 - \pi)^2 \le [\epsilon\Omega^{-2} + \frac{1}{\kappa-\epsilon}]d^2\}$ and $V(x) \ge 2(\Omega^2 + \kappa\epsilon) - 0.5(\Omega^2 +$

$(\kappa+1)\epsilon)(x_1-\pi)^2$ for $x\in M_{d,\pi}$ if $0<\epsilon<\min\{1,\kappa\}$. Note that $M_{d,\pi}\subset M_2$ for $|d|\le 0.5\sqrt{\epsilon}\Omega\frac{\pi}{\zeta}$ and the levels of $V(x)$ are separated into the sets $M_{d,0}$ and $M_{d,\pi}$ if $|d|<\xi$.

Assume that all these restrictions on d are satisfied, i.e. $d<\min\{\Omega^2,\sqrt{\frac{\varrho\lambda_{\min}(Y)}{2}}\frac{\pi}{\zeta},0.5\sqrt{\epsilon}\Omega\frac{\pi}{\zeta},\xi\}$, then by ISS property asymptotically

$$\limsup_{t\to+\infty}|x(t)|_{\mathcal{W}}\le\eta(\|d\|_\infty)$$

for some $\eta\in\mathcal{K}_\infty$ with $\mathcal{W}=\{[0,0]\cup[\pi,0]\}$, and due to compactness of $\mathcal{W}$ the variable $x(t)$ is bounded. By consideration above, asymptotically $x(t)\in M_{d,0}\cup M_{d,\pi}$ and both equilibria $x^a=[\arcsin(d\Omega^{-2}),0]$ and $x^r=[\pi-\arcsin(d\Omega^{-2}),0]$ belong to this intersection (at these steady states $\dot{x}=0$ and, hence, $\dot{V}=0$, then all these points are inside $M_{d,0}\cup M_{d,\pi}$ by construction). Let us show that x^a and x^r are the only invariant solutions in these sets. For this purpose consider

$$\begin{aligned}W(x)&=0.5x_2^2+\sqrt{\Omega^4-d^2}+d[\arcsin(d\Omega^{-2})-x_1]\\&\quad-\Omega^2\cos(x_1),\end{aligned}$$

then $W:\mathbb{R}^2\to\mathbb{R}$ is a C^1 function and $W:M\to\mathbb{R}_+$ is a discontinuous one (depending on the definition of the state space). Note that $W(x^a)=0$ is the global minimum of W in $M\times\mathbb{R}$, and $W(x^r)=2\sqrt{\Omega^4-d^2}+d(2\arcsin(d\Omega^{-2})-\pi)>0$ is a local maximum (W is monotonously growing with respect to $|x_2|$ and $\frac{\partial W}{\partial x_1}=0$ at two points x^a and x^r). For a constant d we obtain

$$\begin{aligned}\dot{W}&=x_2(-\Omega^2\sin(x_1)-\kappa x_2+d)-dx_2+\Omega^2\sin(x_1)x_2\\&=-\kappa x_2^2\le 0.\end{aligned}$$

Let discontinuity of W be at $x_1=0$ and consider the set $M_{d,\pi}$. Recall that x^r is the only steady state contained in $M_{d,\pi}$. Next, we show by contradiction that x^r is the only invariant solution contained in $M_{d,\pi}$. To this end, assume that there is another positively invariant solution $x^*(t)\in M_{d,\pi}$ for all $t\ge 0$. Since $x(t)$ is bounded, W is continuous on $M_{d,\pi}$ with a bounded image and $\dot{W}\le 0$ for all $x\in M_{d,\pi}$, then by LaSalle's invariance principle, $x^*(t)$ has to converge to the largest invariant set contained in $\{x\in M_{d,\pi}:\dot{W}\equiv 0\}$. Clearly, for $x\in M_{d,\pi}$, $\dot{W}\equiv 0$ implies $x(t)=x^r$ for all $t\ge 0$. But x^r is a saddle point and, hence, not attractive. Therefore, x^r is the only invariant solution contained in $M_{d,\pi}$. Consequently, any trajectory $x(t)$ with $x(0)\in M_{d,\pi}\setminus\{x^r\}$ must exit the set $M_{d,\pi}$ at some instant in time (since $\dot{W}<0$ there and x^r is a saddle of W). Outside of $M_{d,0}\cup M_{d,\pi}$ we have $\dot{V}<0$ and it means that the set $M_{d,0}$ is attractive. To analyse the system behaviour inside the invariant set $M_{d,0}$, let the discontinuity of W be at $x_1=\pi$, again since $x(t)$ is bounded, W is continuous on $M_{d,0}$ and $\dot{W}\le 0$, then $x(t)$ has to converge to the largest invariant set into $\dot{W}=0$, that is x^a. □

4.22 A delayed case study

Now consider a time-delay modification of (9):

$$\begin{aligned}\dot{x}_1(t)&=x_2(t),\\\dot{x}_2(t)&=-\Omega^2\sin[x_1(t-\tau)]-\kappa x_2(t)+d(t),\end{aligned}\tag{13}$$

where $\tau>0$ is a fixed delay. The unperturbed system (13) with $d(t)=0$ has the same equilibria as (9), i.e. $[0,0]$ and $[\pi,0]$. The system (13) can be represented as follows:

$$\begin{aligned}\dot{x}_1(t)&=x_2(t),\\\dot{x}_2(t)&=-\Omega^2\sin[x_1(t)]-\kappa x_2(t)\\&\quad+d(t)+\Omega^2\{\sin[x_1(t)]-\sin[x_1(t-\tau)]\}.\end{aligned}$$

By the mean value theorem

$$\begin{aligned}|\sin[x_1(t)]-\sin[x_1(t-\tau)]|&=|\cos[x_1(\phi)]x_2(\phi)\tau|\\&\le|x_2(\phi)|\tau\end{aligned}$$

for some $\phi\in[t-\tau,t]$. Thus, the system (13) can be analysed as a perturbed nonlinear pendulum with part of the input d dependent on the delay. By taking the estimate derived for V in (11) we obtain for $\mu=\epsilon\Omega^{-2}+\frac{1}{\kappa-\epsilon}$:

$$\begin{aligned}D^+V(t)&\le-0.5[\kappa-\epsilon]x_2^2-0.5\epsilon\Omega^2\sin^2(x_1)\\&\quad+\mu\Omega^4x_2^2(\phi)\tau^2+\mu d^2.\end{aligned}$$

It is straightforward to check that

$$\begin{aligned}V(x)&\le 0.5[1+\epsilon]x_2^2+0.5\epsilon\sin^2(x_1)+2[\Omega^2+\kappa\epsilon],\\x_2^2&\le\frac{2}{1-\epsilon}V(x)+\frac{\epsilon}{1-\epsilon}\end{aligned}$$

for $0<\epsilon<\min\{1,\kappa\}$, then for $\rho=\min\{\frac{\kappa-\epsilon}{1+\epsilon},\Omega^2\}$

$$\begin{aligned}D^+V(t)&\le-\rho\{V(t)-2[\Omega^2+\kappa\epsilon]\}\\&\quad+\mu\Omega^4x_2^2(\phi)\tau^2+\mu d^2\\&\le-\rho\{V(t)-2[\Omega^2+\kappa\epsilon]\}\\&\quad+\frac{\mu\Omega^4}{1-\epsilon}\tau^2[2V(\phi)+\epsilon]+\mu d^2.\end{aligned}$$

Therefore,

$$\begin{aligned}V(t)&\ge\frac{6}{\rho}\max\left\{2\frac{\mu\Omega^4}{1-\epsilon}\tau^2|V_t|,2\rho[\Omega^2+\kappa\epsilon]\right.\\&\quad\left.+\frac{\mu\Omega^4}{1-\epsilon}\tau^2\epsilon,\mu d^2\right\}\Rightarrow\\D^+V(t)&\le-0.5\rho V(t)\end{aligned}\tag{14}$$

and V is an ISS–LR function for (13), provided that

$$\frac{12}{\rho}\frac{\mu\Omega^4}{1-\epsilon}\tau^2 < 1. \tag{15}$$

The inequality (15) is a delay-dependent stability condition for (13), which is always satisfied for a sufficiently small delay τ. The set of asymptotic attraction for (13) can be evaluated from (14).

Remark 4.1: If we assume that $\max\{0, \frac{\kappa-\Omega^2}{1+\Omega^2}\} < \epsilon < \min\{1, \kappa\}$, then $\min\{\frac{\kappa-\epsilon}{1+\epsilon}, \Omega^2\} = \frac{\kappa-\epsilon}{1+\epsilon}$ and the condition (15) can be rewritten as follows:

$$\tau^2 < \frac{1}{12\Omega^2}\frac{1-\epsilon}{1+\epsilon}\frac{1}{\epsilon(\kappa-\epsilon)+\Omega^2}.$$

Since the functions $\frac{1-\epsilon}{1+\epsilon}$ and $\frac{1}{\epsilon(\kappa-\epsilon)+\Omega^2}$ are decreasing for $\epsilon \in (\max\{0, \frac{\kappa-\Omega^2}{1+\Omega^2}\}, \min\{1,\kappa\})$, selecting $\epsilon = \max\{0, \frac{\kappa-\Omega^2}{1+\Omega^2}\} + \varepsilon$ for a sufficiently small $\varepsilon > 0$ optimises the value of the admissible delay τ to

$$\tau^* = \frac{1}{2\Omega}\sqrt{\frac{1-\epsilon}{1+\epsilon}\frac{1/3}{\epsilon(\kappa-\epsilon)+\Omega^2}},$$

i.e. for any $\tau < \tau^*$ the system (13) admits V as an ISS–LR function.

5. Application to a microgrid composed of two droop-controlled inverters with delay

In this section, the theoretical results of Section 3 are applied to our main motivating application: a droop-controlled microgrid with delays. In particular, we are interested in conditions for ISS of such systems. In order to tackle this problem, we proceed along the lines detailed in Section 4. The analysis is conducted under a reasonable assumption of constant voltage amplitudes. Then, a lossless droop-controlled microgrid formed by two inverters with delay can be modelled as Schiffer et al. (2014):

$$\begin{aligned}\dot\theta(t) &= \omega_1(t) - \omega_2(t),\\ \tau_{P_1}\dot\omega_1(t) &= -\omega_1(t) - k_{P_1}a_{12}\sin[\theta(t-\tau_{d_1})]\\ &\quad + c_1 + d_1(t),\\ \tau_{P_2}\dot\omega_2(t) &= -\omega_2(t) + k_{P_2}a_{12}\sin[\theta(t-\tau_{d_2})] + c_2\\ &\quad + d_2(t),\end{aligned} \tag{16}$$

where $\theta(t) \in [0, 2\pi)$ is the phase difference in inverters, $\omega_1(t), \omega_2(t) \in \mathbb{R}$ are time-varying frequencies of the inverters; $\tau_{d_1} > 0$ and $\tau_{d_2} > 0$ are delays caused by the digital controls required to implement the droop controls; $\tau_{P_1} > 0$, $\tau_{P_2} > 0$, $k_{P_1} > 0$, $k_{P_2} > 0$, $a_{12} > 0$, c_1 and $c_2 = -\frac{k_{P_2}}{k_{P_1}}c_1$ are constant parameters, the disturbances $d_1(t)$ and $d_2(t)$ represent additional model uncertainties. We say that a solution of (16) is phase-locked if $\theta(t) = \theta_0$ is constant $\forall t \in \mathbb{R}_+$ for some $\theta_0 \in [0, 2\pi)$ (Franci, Chaillet, Panteley, & Lamnabhi-Lagarrigue, 2012). If this property holds asymptotically, *i.e.* for $t \to +\infty$, we speak about an asymptotic phase-locking.

For brevity of presentation, we impose the following restrictions on the values of parameters.

Assumption 5.1: $\tau_{P_1} = \tau_{P_2} = \tau_P > 0$ *and* $\tau_{d_1} = \tau_{d_2} = \tau > 0$.

Under this assumption, define the new coordinates:

$$x_1 = \theta,\ x_2 = \omega_1 - \omega_2,\ x_3 = \frac{k_{P_2}}{k_{P_1}}\omega_1 - \omega_2,$$

then the system (16) can be rewritten as follows:

$$\dot x_1(t) = x_2(t), \tag{17}$$

$$\begin{aligned}\tau_P\dot x_2(t) &= -x_2(t) - [k_{P_1} + k_{P_2}]a_{12}\sin[x_1(t-\tau)]\\ &\quad + \left[1 + \frac{k_{P_2}}{k_{P_1}}\right]c_1 + d_1 - d_2,\end{aligned} \tag{18}$$

$$\tau_P\dot x_3(t) = -x_3(t) + \frac{k_{P_2}}{k_{P_1}}d_1 - d_2. \tag{19}$$

Thus, the system (16) is decomposed into two independent subsystems: (17), (18) and (19). The variable x_3 converges asymptotically to zero with the time constant τ_P if $d_1 = d_2 = 0$. Hence, asymptotically the frequencies ω_1 and ω_2 are locked. The dynamics (17) and (18) have the form of (13) for $d = [1 + \frac{k_{P_2}}{k_{P_1}}]c_1 + d_1 - d_2$ and, as it has been established earlier, have pAG and pGS properties from Definitions 3.1 and 3.2, respectively, if condition (15) is satisfied, which for (17) and (18) takes the form:

$$\tau^2 < \frac{\min\left\{\frac{\tau_P^{-1}-\epsilon}{1+\epsilon}, \frac{[k_{P_1}+k_{P_2}]a_{12}}{\tau_P}\right\}}{12\frac{[k_{P_1}+k_{P_2}]^2a_{12}^2}{\tau_P^2(1-\epsilon)}\left[\frac{\epsilon}{[k_{P_1}+k_{P_2}]a_{12}} + \frac{1}{1-\tau_P\epsilon}\right]} \tag{20}$$

for $0 < \epsilon < \min\{1, \tau_P^{-1}\}$. Therefore, for a sufficiently small delay τ the inverters will demonstrate a phase-locking behaviour. According to Nussbaumer et al. (2008), a good estimate of the overall delay introduced by the digital control is $\tau = 1.75T_S$[1], where $T_S = 1/f_S$ and $f_S \in \mathbb{R}_{>0}$ is the switching frequency of the inverter. Since usually $f_S \in [5, 20]$ kHz (Green & Prodanovic, 2007), τ is reasonably small in most practical applications. Hence, we expect condition (20) to be satisfied for most practical choices of parameters τ_P, k_{P_1} and k_{P_2}.

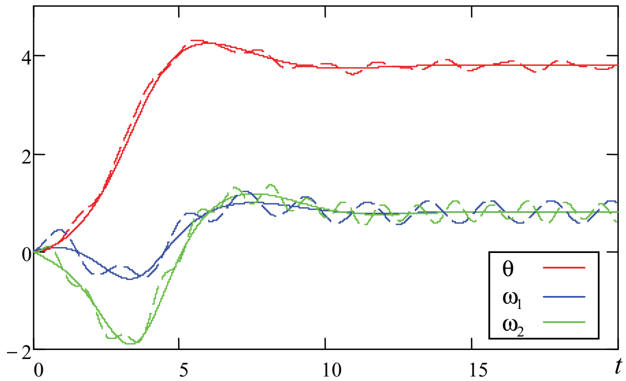

Figure 1. Simulation results for the system (16). The solid lines show the state trajectories for the case $d_1(t) = d_2(t) = 0$. The dashed lines correspond to the case $d_1(t) = 0.8\sin(3t)$, $d_2(t) = 0.9\sin(5t)$.

The analysis is illustrated in a simulation example with the following set of parameters for the system (16): $\tau_P = 1$, $k_{P_1} = 10$, $k_{P_2} = 20$, $a_{12} = 0.1$, $c_1 = 0.2$ and $\tau = 0.05$. Condition (20) is satisfied for $\epsilon = 0.5\min\{1, \tau_P^{-1}\}$. The simulation results are shown in Figure 1. The solid lines represent the state $(\theta, \omega_1, \omega_2)^T$ trajectories for the case $d_1(t) = d_2(t) = 0$, and the dashed lines correspond to $d_1(t) = 0.8\sin(3t)$, $d_2(t) = 0.9\sin(5t)$. The phase-locking phenomenon is observed in these simulation results.

6. Conclusions

Sufficient conditions for ISS of multistable systems with delay have been derived. The conditions have been established using Lyapunov–Razumikhin functions. The potential of the presented approach has been illustrated by exhibiting several new robustness properties for a nonlinear pendulum with delay. Furthermore, it has been shown that asymptotic phase-locking in a lossless droop-controlled microgrid formed by two inverters with delays can be analysed based on a perturbed pendulum model. By exploiting this fact, a delay-dependent condition for ISS of such a microgrid has been presented.

Future work will consider an extension of the analysis to more complex inverter models with delays (e.g. time-varying voltages or internal filter and controller dynamics).

Note

1. The overall delay reduces to $\tau = 1.5T_S$ if no moving average function for the measurement is used Nussbaumer et al. (2008).

Acknowledgements

This work was supported in part by the Government of Russian Federation (Grant 074-U01) and the Ministry of Education and Science of Russian Federation (Project 14.Z50.31.0031).

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

Government of Russian Federation [Grant Number 074-U01]; Ministry of Education and Science of Russian Federation [Project Number 14.Z50.31.0031].

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