Finite-Time Estimation of Time-Varying Frequency Signals in Low-Inertia Power Systems

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Abstract— **The ongoing and unprecedented transformation of power systems leads to a reduction in the number of conventional power plants, which are the classical actuators of the grid. In addition, this development results in a decreasing system inertia, which is expected to yield faster frequency dynamics. Therefore power-electronics-interfaced units have to take over system control tasks and, in particular, frequency control. For this purpose, accurate and fast estimation algorithms for time-varying frequency signals are needed. Motivated by this fact, we propose a time-varying parameter estimator and a tuning criterion, which for sufficiently small initial estimation errors allows to reconstruct the time-varying frequency signal of a symmetric three-phase waveform in finite time. The proposed estimator is derived by using a time-varying version of the super twisting algorithm and its performance is illustrated via numerical examples.**

I. Introduction

A. Motivation

The steady uprise of renewable energy sources in power systems worldwide is expected to drastically impact the dynamics and, consequently, the control of future power systems [1], [2]. A main reason for this is that most renewable generation units are interfaced to the network via power electronics, instead of via synchronous generators (SGs) as employed in conventional power plants. Hence, the current developments result in a decreasing amount of rotational inertia. This in turn is expected to yield faster and more volatile system dynamics [2].

In addition, for decades SGs have been the main actuators used to ensure power system performance by providing the necessary system services. Yet, as conventional SG-interfaced units are being faded out, inverter-interfaced units need to take over system control tasks [2]. Amongst these, frequency regulation is one of the most important operational objectives in any AC power system [3].

Clearly, a fundamental prerequisite for a fast and efficient deployment of frequency control action via power converters is the availability of an accurate measurement of the (local) electrical frequency. Unlike in SGs, in power inverters there are no rotational elements allowing to measure the angular speed. Therefore, the frequency needs to be estimated from the available AC measurements at the inverter terminals. The standard estimation algorithms used for this purpose are called phase-locked-loops (PLLs) [4], [5]. There exist several PLL schemes and most of them share the property that a linear control is used to track the frequency signal, such as the popular PI-based synchronous reference frame PLL (SRF-PLL) [4], [5]. Yet, it has been widely recognized that this approach exhibits severe limitations in the presence of fast frequency variations [6] and high deviations from the nominal frequency [7]. Furthermore, it has been shown in several studies [8], [9], [10] that the PLL dynamics have a significant impact on the closed-loop performance of inverter-based frequency control strategies. These problematics in turn have motivated the development of new techniques. Amongst these are methods based on series expansion [11], [12], least squares [13], adaptive filtering [14], [15], gradient methods [16], and sliding mode control [17], [18].

In conventional power systems formed by bulk SGs the assumption of having a stiff grid (i.e., a voltage waveform with a constant frequency) at the inverter's point of connection was acceptable. Yet, as outlined above, future power systems are expected to exhibit larger and faster frequency variations. Hence, in order to provide a swift and reliable frequency estimate in such a setting, an estimation algorithm capable of tracking time-varying signals and exhibiting fast recovery capabilities is required. This motivates the present work.

B. Contributions

In light of the above discussion, the main contributions in this paper are:

1) To provide a frequency estimator, which is capable of exactly tracking a time-varying frequency signal of a symmetric three-phase AC waveform. This is achieved by interpreting the frequency as a time-varying parameter and, inspired by ideas from [19] and [20], deriving a time-varying version of the super twisting algorithm in vector form for the estimator design. The present approach avoids the typical over-parameterization pertinent to methods based on series expansion [21], [22], [23] and does, hence, not increase the dimension of the frequency estimator.
2) To provide a set of sufficient conditions for the estimator gains, the feasibility of which guarantees exact convergence of the estimated frequency signal in finite time.
3) To demonstrate the improved performance of the proposed algorithm with respect to the standard SRF-PLL

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[4], [5] via several numerical examples.

Notation: Let $\mathbb{R}$ denote the set of real numbers. For $a \in \mathbb{R}$, $\mathbb{R}_{\geq a}$ denotes the open interval $[a, \infty)$. $\mathbb{R}^n$ denotes the set of n-dimensional real-valued vectors and $\mathbb{R}^{n\times m}$ denotes the set of real-valued $n \times m$ matrices. Let $\mathbf{I}_n$ denote the identity matrix of dimension n. For $A \in \mathbb{R}^{n\times m}$, $A^\top$ denotes the transpose of A. For $A \in \mathbb{R}^{n\times n}$ with $A = A^\top$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and the largest eigenvalues of A, respectively. For a vector $\nu \in \mathbb{R}^n$, $\|\nu\| = \sqrt{\nu^\top \nu}$ represents its Euclidean norm and its 1-norm is given by $\|\nu\|_1 = \sum_{i=1}^n |\nu_i|$. For $A = A^\top$, $A > 0$ means that it is a positive definite matrix, whereas $A \geq 0$ means that A is positive semi-definite. Consider a block-defined matrix $A \in \mathbb{R}^{(n+m)\times(n+m)}$ described by

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \begin{matrix} A_{11} \in \mathbb{R}^{n\times n}, & A_{12} \in \mathbb{R}^{n\times m}, \\ A_{21} \in \mathbb{R}^{m\times n}, & A_{22} \in \mathbb{R}^{m\times m}. \end{matrix}$$

Then the Schur complements of A are denoted by $A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$ and $A/A_{22} = A_{11} - A_{12}A_{22}^{-1}A_{21}$.

II. Problem statement

We consider a symmetric three-phase AC signal $v : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ with constant amplitude $A > 0$ described by

$$v(t) = A\,\left[\cos\left(\phi(t)\right),\, \cos\left(\phi(t) - 2/3\pi\right),\, \cos\left(\phi(t) + 2/3\pi\right)\right]^\top, \quad (1)$$

where $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ represents the instantaneous phase of the signal. In applications, the signal $v(t)$ corresponds to the three-phase voltage waveform at the generation unit's point of connection to the grid, see [4].

It is assumed that $\phi(t)$ is a smooth monotonically increasing function of the time t. Its time derivative $\dot{\phi}(t) = \omega(t)$ (i.e., the instantaneous frequency) is assumed to be a Lipschitz function of time with Lipschitz constant $\Delta^\star > 0$. This means that the following inequality holds:

$$|\omega(t_1) - \omega(t_2)| \leq \Delta^\star |t_2 - t_1|, \; \forall\, t_1 \in \mathbb{R}_{\geq 0}, \; \forall\, t_2 \in \mathbb{R}_{\geq 0}. \quad (2)$$

Given that $\phi(t)$ is assumed to be monotonically increasing, we have that $\omega(t) \geq 0$ for all $t \in \mathbb{R}_{\geq 0}$.

It is well-known that symmetric three-phase signals can be equivalently represented by two quantities [4], [24]. We use this fact and consider the $\alpha\beta$-transformation [4] with transformation matrix $T \in \mathbb{R}^{2\times 3}$,

$$T = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ 0 & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}.$$

By applying T to $v(t)$ defined in (1), a reduced set of signals can be obtained, i.e.,

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = T\,v(t) = A \begin{bmatrix} \cos\left(\phi(t)\right) \\ \sin\left(\phi(t)\right) \end{bmatrix}.$$

The behavior of $y(t)$ over time can then be described by the following nonlinear dynamical model:

$$\dot{y}(t) = \omega(t) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} y(t) = \omega(t) \begin{bmatrix} -y_2(t) \\ y_1(t) \end{bmatrix}. \quad (3)$$

The main purpose of this note is to design a frequency estimator, which is capable of exactly estimating the instantaneous time-varying frequency $\omega(t)$ in finite time. To this end, the assumptions below are employed in the present work:

Assumption 1. *The signals $\phi(t)$ and $\omega(t)$ have the following properties:*

1) *The constant amplitude $A > 0$ is known.*
2) *The signal $\phi(t)$ is monotonically increasing.*
3) *The inequality* (2) *holds and a constant $\Delta \geq \Delta^\star$ is known.*
4) *The variable $v(t)$ is available as a measurement.*
5) *The frequency $\omega(t)$ has an upper bound such that the time period between sign changes of both $y_1(t)$ and $y_2(t)$ is lower bounded by some constant $h > 0$.*

Remark 1. Given that $(y_1^2(t) + y_2^2(t))^{\frac{1}{2}} = A$, Assumption 1.1 is not restrictive. Assumption 1.2 implies that there is no negative frequency, which is reasonable given the underlying physical setting of the considered problem. Assumption 1.3 is a technical assumption that allows to give a dynamical representation for $\omega(t)$ as is shown in the next section. Assumption 1.4 implies that $y(t)$ is known and follows the dynamics in (3). Finally, Assumption 1.5 is a technical condition that originates from the proof.

III. Preliminaries

The problem of estimating $\omega(t)$ from (3) closely resembles the classical problem of estimating constant parameters [25], [26] from the dynamics

$$\dot{z}(t) = b^\top(t)\theta, \quad (4)$$

where $z : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, is a measured signal, $b : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times m}$, represents the regressor and $\theta \in \mathbb{R}^m$ is the vector of unknown constant parameters. In the classical problem statement [25], [26] the number of measurements $z(t)$ is, usually, less than the number of parameters, i.e., $n < m$. The estimation is then possible when the parameters are constant and if the regressor $b(t)$ satisfies a certain persistent of excitation condition [26, Eq. 3.5]. The main difference between the classical problem and the problem posed in this note is that the frequency $\omega(t)$, i.e., the unknown quantity of interest, is not constant. Therefore, the classical estimation methods are not applicable.

As a consequence, a natural step in the present case is to interpret $\omega(t)$ as a *time-varying* parameter. In the case of time-varying parameters, an important fact is that the standard persistent of excitation is not sufficient for the parameter reconstruction [19], [27]. The main reason for this is briefly sketched below.

Consider again (4), but assume that the parameters θ are time-varying. Also assume that one has $\dot{z}(t)$ at hand. Then (4) can be seen as an algebraic relation between known signals and the time-varying parameters. Under these circumstances, the necessary and sufficient condition for reconstructing the parameters is that the regressor is uniformly injective [27, Theo. 4.1]. This means that at least the same number of measurements as parameters are needed. Fortunately, this condition is satisfied in the problem posed in Section II. To see this, recall that in (3) the regressor can be taken as

$$b^\top(t) = [b_1(t),\, b_2(t)] = [-y_2(t),\, y_1(t)]. \quad (5)$$

Since $b^\top(t)b(t) = A^2 > 0$, the regressor is uniformly injective. Hence, it is, in principle, possible to reconstruct the frequency $\omega(t)$.

After having established that it is possible to reconstruct $\omega(t)$, we seek to impose a dynamical model for its behavior. This is an essential prerequisite to be able to derive a dynamical system (observer) for its reconstruction. Since by Assumption 1.3, $\omega(t)$ is a Lipschitz function of time, it is also absolutely continuous, which implies the existence of a function $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}$, such that $\omega(t) = \omega(t_0) + \int_{t_0}^{t} \xi(s)\mathrm{d}s$, or, equivalently, $\dot{\omega}(t) = \xi(t)$ almost everywhere. Furthermore, the Lipschitz condition also means that $\xi(t)$, where defined, is bounded by Δ, i.e., $|\xi(t)| \leq \Delta$. By defining the regressor as $b^\top(t) = [-y_2(t),\, y_1(t)]$ and using the regressor together with the *model* introduced for $\omega(t)$, we can extend the dynamics (3) to

$$\begin{aligned} \dot{y}(t) &= b(t)\omega(t), \\ \dot{\omega}(t) &= \xi(t), \end{aligned} \tag{6}$$

where $\xi(t)$ is treated as an unknown, bounded disturbance. The model (6) will be at the core of the estimation strategy proposed in the next section.

IV. MAIN RESULT

A time-varying version of the super twisting algorithm, capable of estimating a time-varying parameter, has been proposed in [19]. The algorithm in [19] exploits a similar structure to the model (6) with an important difference, namely the number of available measurements. Thanks to the additional measurement available in the present case, the restriction that appears on the disturbance in [19, Asm. 1] can be relaxed. However, unfortunately, this extra measurement makes a direct application of the algorithm in [19] impossible. To circumvent this obstacle, we propose to combine a vector form of the super twisting algorithm, introduced in [20], with the time-varying approach of [19]. This results in the following time-varying parameter estimator:

$$\begin{aligned} \dot{\hat{y}}(t) &= -k_1 \frac{\hat{y}(t) - y(t)}{\|\hat{y}(t) - y(t)\|^{\frac{1}{2}}} + b(t)\hat{\omega}(t), \\ \dot{\hat{\omega}}(t) &= -k_2 b^\top(t) \frac{\hat{y}(t) - y(t)}{\|\hat{y}(t) - y(t)\|}, \end{aligned} \tag{7}$$

where $\hat{y}(t)$ and $\hat{\omega}(t)$ represent the estimates of $y(t)$ and $\omega(t)$, respectively, and the constants $k_1 > 0$ and $k_2 > 0$ are the gains of the algorithm. The nonlinear terms in the algorithm (7) consist of

$$\psi_1(\nu) := \frac{\nu}{\|\nu\|^{\frac{1}{2}}} \quad \text{and} \quad \psi_2(\nu) := \frac{\nu}{\|\nu\|},\ \nu \in \mathbb{R}^2.$$

These functions correspond to the vector form of the scalar functions $|\cdot|^{\frac{1}{2}}\mathrm{sign}(\cdot)$ and $\mathrm{sign}(\cdot)$. For this reason, ψ_1 is defined as zero when its argument is zero, whereas ψ_2 has a bounded discontinuity in this point.

For the frequency estimator described in (7), we have the following result:

Theorem 1. *Consider* (7) *together with Assumption 1. Let* $c > 0$ *and set the estimator gains as*

$$\begin{aligned} k_1 &= \left(\frac{1}{4} + \sqrt{2}\right) A + c, \\ k_2 &= \frac{9(5+\sqrt{2})A}{8\,c} + \frac{9+40\sqrt{2}}{8} + \frac{5\,c}{2\,A} + \frac{\sqrt{2}\Delta}{c} + \frac{(1+\sqrt{2})\Delta^2}{\sqrt{2}\,A\,c}. \end{aligned} \tag{8}$$

Define the positive real parameters

$$\eta := 1 + \frac{2 + 2\big(2\,k_2 - k_1/A - 1\big)^{\frac{1}{2}}}{2\,k_2 - k_1/A - 1}, \tag{9}$$

$$\begin{aligned} \lambda_\pm := {} & \frac{A(1+2\,k_2) - k_1}{2\,A} \\ & \pm \frac{\big(k_1^2 + 2\,A\,k_1(1-2\,k_2) + A^2(9 + 4\,k_2(k_2-1))\big)^{\frac{1}{2}}}{2\,A}, \end{aligned} \tag{10}$$

and let $e_0 = \hat{y}(t_0) - y(t_0)$, $x_0 = \hat{\omega}(t_0) - \omega(t_0)$ *be the initial errors. Then, if the frequency estimator* (7) *is initialized such that*

$$\frac{1}{8} A\, h \geq \left(\frac{\lambda_+}{\lambda_-}\left(\|e_0\| + x_0^2\right)\right)^{\frac{1}{2}} \left(1 - \frac{1}{\eta^{\frac{1}{2}}}\right), \tag{11}$$

$\hat{\omega}(t)$ *converges exactly to* $\omega(t)$ *in finite time.*

Remark 2. For the given definitions of k_1 and k_2 and since $c > 0$, it can be shown in a straightforward manner that η in (9) is well defined.

Theorem 1 provides a set of sufficient conditions to ensure the convergence of the estimation error. These conditions (8) represent lower bounds on the estimator gains k_1 and k_2. In addition, these bounds depend on the signal amplitude A and the Lipschitz constant for the time derivative of the frequency Δ. Furthermore, condition (11) imposes a restriction on the initial conditions and the minimum time h between sign switches. For a small enough initial error, or equivalently, for large enough h, the finite-time convergence can be ensured. The parameters λ_-, λ_+ and η, which all depend on k_1 and k_2, provide some degree of freedom to ensure (11).

It is evident that (11) can not be satisfied for any arbitrary set of initial errors e_0, x_0 and any h. Therefore, the result of Theorem 1 is a local convergence result. However, we emphasize that the condition (11) appears because in the proof of Theorem 1 a switched Lyapunov function is used. Each switch induces a growth in the value of the Lyapunov function, and this growth, at the same time, requires to impose (11) in order to ensure that in average the Lyapunov function decreases. For this reason, it is the belief of the authors that condition (11) is not essential for the actual estimator convergence, but rather a consequence of the employed technique of proof. In addition, for the considered application in this paper, i.e., frequency estimation in power systems, the condition (11) can be expected to hold in most practical scenarios. This is also demonstrated via numerical examples in Section VI.

In the next section, we present the convergence analysis and the proof of Theorem 1. As outlined previously, the analysis is performed by using a switched Lyapunov function. This strategy corresponds to the one used in [19], and with it, we recover the main properties of the algorithm introduced in the aforementioned reference.

V. CONVERGENCE ANALYSIS AND PROOF OF THEOREM 1

The study of the algorithm convergence is equivalent to the study of the stability of its error dynamics. With the purpose of finding the error dynamics, define $e(t) = \hat{y}(t) - y(t)$ and

i	1	2	3	4
sign$(b_1(t))$	-1	1	1	-1
sign$(b_2(t))$	1	1	-1	-1
$\mathbf{p}_{i1}$	1	-1	-1	1
$\mathbf{p}_{i2}$	-1	-1	1	1

TABLE I

DEFINITION OF $\mathbf{p}_i$ IN TERMS OF THE SIGN OF $b(t)$.

$x(t) = \hat{\omega}(t) - \omega(t)$ as error variables. The time derivative of them yields the error dynamics

$$\begin{aligned} \dot{e}(t) &= -k_1\psi_1\big(e(t)\big) + b(t)x(t), \\ \dot{x}(t) &= -k_2 b^\top(t)\psi_2\big(e(t)\big) - \xi(t). \end{aligned} \tag{12}$$

Since the dynamics of $x(t)$ has a discontinuity in the set $e = 0$, the solutions of (12) have to be understood in the sense of Filippov [28]. To analyze the stability of the solution $\{e(t) = 0, x(t) = 0\}$, we follow [19] and use multiple Lyapunov function candidates depending on the sign of $b(t)$. Such approach is totally analogous to standard methods used to study the stability of switched systems [29]. The structure of the employed Lyapunov function candidates is as follows:

$$V_i\big(e, \tilde{\omega}\big) = \zeta^\top P_i \zeta, \quad \zeta := \begin{bmatrix} \psi_1(e) \\ x(t) \end{bmatrix}, \quad P_i = \begin{bmatrix} p_1\mathbf{I}_2 & \mathbf{p}_i \\ \mathbf{p}_i^\top & p_2 \end{bmatrix},$$

for $i = \{1, 2, 3, 4\}$, and positive design parameters p_1 and p_2. Let $\mathbf{p}_i^\top = [\mathbf{p}_{i1}, \mathbf{p}_{i2}]$ and $b^\top(t) = [b_1(t), b_2(t)]$. The value of $\mathbf{p}_i$ depends on the sign of $b(t)$ as it is shown in Table I. Hence, P_i is positive definite iff $p_1 p_2 > 2$. By choosing $p_1 > 0$, this can be easily verified from the Schur complement $p_2 - 2p_1^{-1} > 0$. Each V_i is a valid Lyapunov function candidate because each is a quadratic form of ζ and $\|\zeta\| \to \infty$ when $\|\{e, x\}\| \to \infty$.

The idea of the proof is to show that the time derivative of each V_i is negative definite during the period of time where the sign of $b(t)$ corresponds to the i-th interval, see Table I, and thus the estimation error decrease during that time. However, since each V_i defines different level sets, it is also necessary to analyze how these level sets intersect each other in each change of Lyapunov function. For these reasons, the proof is split into two steps: the decrease analysis of the Lyapunov functions V_i and the effect of the change of the Lyapunov function.

A. Decrease of the error between sign changes

The time derivative of each V_i can be computed as $\dot{V}_i(t) = \zeta^\top(t)P_i\dot{\zeta}(t) + \dot{\zeta}^\top(t)P_i\zeta(t)$. The time derivative of $\zeta(t)$ (where it exists) can be expressed in the following manner:

$$\begin{aligned} \dot{\zeta}(t) &= \frac{1}{2\,\|e(t)\|^{\frac{1}{2}}} A(e,t)\zeta(t) + \Xi(t), \\ A(e,t) &= \begin{bmatrix} -k_1 G(e) & G(e)b(t) \\ -2k_2 b^\top(t)G(e) & 0 \end{bmatrix}, \end{aligned}$$

where $G(e) = 2\mathbf{I}_2 - e\,e^\top/\|e\|^2$, and $\Xi^\top(t) = [0, -\xi(t)]$. Then, the derivative of each V_i results in

$$\begin{aligned} \dot{V}_i(t) = &\frac{1}{\|e(t)\|^{\frac{1}{2}}}\zeta^\top(t)\left(P_i A(e,t) + A^\top(e,t)P_i\right)\zeta(t) \\ &+ 2\,\zeta^\top(t)P_i\Xi(t). \end{aligned} \tag{13}$$

By noting that $G(e) = G^\top(e)$ as well as $G(e)\psi_1(t) = \psi_1(e)$, and defining the matrix $Q(e,t)$

$$Q(e,t) = \begin{bmatrix} Q_{11}(e,t) & Q_{12}(e,t) \\ Q_{12}^\top(e,t) & q_{22}(e,t) \end{bmatrix},$$

where

$$\begin{aligned} Q_{11}(t) &= 2\,k_1 p_1 \mathbf{I}_2 + 2\,k_2\big(\mathbf{p}_i b^\top(t) + b(t)\mathbf{p}_i^\top\big), \\ Q_{12}(t) &= k_1\mathbf{p}_i + \big(2\,k_2 p_2 - p_1\big)b(t), \\ q_{22}(e,t) &= -\mathbf{p}_i^\top G(e)b(t) - b^\top(t)G(e)\mathbf{p}_i, \end{aligned}$$

we can rewrite (13) as

$$\dot{V}_i(t) = -\frac{\zeta^\top(t)Q(e,t)\zeta(t)}{2\,\|e(t)\|^{\frac{1}{2}}} - 2\,\xi(t)(\psi_1^\top(e(t))\mathbf{p}_i + p_2 x(t)). \tag{14}$$

Now we proceed to bound the disturbance term. Given the assumed bound on $\xi(t)$, we have

$$\frac{4\,\Delta}{2\,\|e(t)\|^{\frac{1}{2}}}\left(\sqrt{2}\psi_1^\top(e(t))\psi_1(e(t)) + p_2\|e(t)\|^{\frac{1}{2}}|x(t)|\right) \geq 2\,|\xi(t)|\big|\psi_1^\top(e(t))\mathbf{p}_i + p_2 x(t)\big|,$$

where we have used the fact that $\|\psi_1(e(t))\|^2 = \|e(t)\|$. Let $\epsilon_1 > 0$. By Young's inequality we have

$$2\,p_2\Delta\|e(t)\|^{\frac{1}{2}}|x(t)| \leq 2\frac{p_2^2\Delta^2}{\epsilon_1}\|e(t)\| + \frac{\epsilon_1}{2}x^2(t).$$

Hence, the disturbance term can be bounded as

$$\frac{1}{2\,\|e(t)\|^{\frac{1}{2}}}\zeta^\top(t)R\,\zeta(t) \geq 2\,\xi(t)(\psi_1^\top(e(t))\mathbf{p}_i + p_2 x(t)), \tag{15}$$

with $R = \operatorname{diag}\{4(\sqrt{2}\Delta + p_2^2\Delta^2/\epsilon_1)\mathbf{I}_2,\, \epsilon_1\}$. Now, by combining (14) and (15) we obtain the following bound for the time derivative of V_i: $\dot{V}_i(t) \leq -\zeta^\top(t)\big(Q(e,t) - R\big)\zeta(t)/(2\,\|e(t)\|^{1/2})$. Additionally, we are interested in finding $\gamma > 0$ such that $Q(e,t) - R \leq \gamma\,P_i$. This will imply that

$$\dot{V}_i(t) \leq -\frac{\gamma}{2\,\|e(t)\|^{\frac{1}{2}}}V_i(t). \tag{16}$$

Hence, we need to find positive constant values of p_1, p_2, k_1, k_2, ϵ_1 and γ such that the matrix $Q(e,t) - R - \gamma\,P_i$ is positive definite.

Define the matrix $\Gamma(t) = -\mathbf{p}_i b^\top(t) - b(t)\mathbf{p}_i^\top$. Recall the regressor $b(t)$ defined in (5) and the definition of $\mathbf{p}_i$ in Table I. Then, computing the eigenvalues of $\Gamma(t)$ yields $\lambda_{1,2}\big(\Gamma(t)\big) = \|y(t)\|_1 \pm A\sqrt{2}$. Furthermore, $q_{22}(e,t)$ can be written as

$$q_{22}(e,t) = 4\,\|y(t)\|_1 - \frac{1}{\|e(t)\|^2}e^\top(t)\Gamma(t)e(t) \geq \big(3 - \sqrt{2}\big)A.$$

Thus, we have $q_{22}(e,t) - R_{22} - \gamma\,\mathbf{p}_{22} \geq (3 - \sqrt{2})A - \epsilon_1 - \gamma\,p_2$. Set $\epsilon_1 = (2 - \sqrt{2})A$, then $q_{22}(e,t) - R_{22} - \gamma\,\mathbf{p}_{22} \geq A - \gamma\,p_2$. Now, the Schur complement of $(Q(e,t) - R - \gamma\,P_i)/(q_{22}(e,t) - R_{22} - \gamma\,\mathbf{p}_{22})$ can be bounded as:

$$Q_{11}(t) - R_{11} - \gamma\,p_1\mathbf{I}_2 - \frac{\big\|Q_{12}(t) - \gamma\,\mathbf{p}_i\big\|^2}{A - \gamma\,p_2}\mathbf{I}_2.$$

By noting that $\mathbf{p}_i = -\text{sign}(b(t))$ in each interval we have

$$Q_{12}(t) = \begin{bmatrix} \text{sign}(y_2(t))\big(k_1 - (2\,k_2 p_2 - p_1)|y_2(t)|\big) \\ -\text{sign}(y_1(t))\big(k_1 - (2\,k_2 p_2 - p_1)|y_1(t)|\big) \end{bmatrix}.$$

By choosing $p_1 = 2\,k_2 p_2 - k_1/A$, $Q_{12}(t)$ becomes

$$Q_{12}(t) = \begin{bmatrix} \operatorname{sign}(y_2(t))k_1\big(1-|y_2(t)|/A\big) \\ -\operatorname{sign}(y_1(t))k_1\big(1-|y_1(t)|/A\big) \end{bmatrix},$$

with $\|Q_{12}(t)\|^2 \le 2\,k_1^2$. Note that $\|Q_{12}(t) - \gamma \mathbf{p}_i\|^2 \le 4(k_1^2 + \gamma^2)$. By using these inequalities, the Schur complement of $Q(t)$ can be bounded from below by

$$2\,k_1\left(2\,k_2p_2 - \frac{k_1}{A}\right)\mathbf{I}_2 - \gamma\left(2\,k_2p_2 - \frac{k_1}{A}\right)\mathbf{I}_2 - 4\sqrt{2}A\,k_2\mathbf{I}_2 \\ - 4\left(\sqrt{2}\Delta + \frac{\Delta^2 p_2^2}{\sqrt{2}A(\sqrt{2}-1)}\right)\mathbf{I}_2 - 4\frac{k_1^2+\gamma^2}{A-\gamma\,p_2}\mathbf{I}_2.$$

The above expression can be made non-negative if

$$\gamma\,p_2 \le A, \quad k_1 \ge \sqrt{2}\frac{A}{p_2} + \frac{\gamma}{2}, \quad k_2 \ge \frac{n(k_1,\gamma,\Delta,A,p_2)}{d(k_1,\gamma,A,p_2)}, \tag{17}$$

where

$$n(k_1,\gamma,\Delta,A,p_2) = 4\,A(\sqrt{2}-1)\big(k_1^2+\gamma^2\big) + \big(A-\gamma\,p_2\big)\times \\ \times\Big(2\sqrt{2}\Delta\big(\Delta\,p_2^2 + 2(\sqrt{2}-1)A\big) + (\sqrt{2}-1)(2\,k_1-\gamma)k_1\Big), \\ d(k_1,\gamma,A,p_2) = 2A\big(\sqrt{2}-1\big)\big(A-\gamma p_2\big)\big(p_2(2k_1-\gamma) - 2\sqrt{2}A\big).$$

If k_1, k_2, p_2 and γ are chosen such that the previous inequalities are satisfied, we otbtain inequality (16). Furthermore, by noting that $\|e(t)\|^{1/2} \le \|\zeta(t)\| \le V_i^{1/2}(t)/\lambda_{\min}^{1/2}(P_i)$, we can write (16) as a differential inequality of V_i:

$$\dot{V}_i(t) \le -\alpha\,V_i^{\frac{1}{2}}(t);\ \alpha := \frac{1}{2}\gamma\,\lambda_{\min}^{\frac{1}{2}}(P_i). \tag{18}$$

By the Comparison Lemma [30] and using separation of variables, we have

$$V_i^{\frac{1}{2}}(t) \le \max\left\{V_i^{\frac{1}{2}}(t_0) - \frac{\alpha}{2}\big(t-t_0\big),\, 0\right\}. \tag{19}$$

This allows to estimate the decrease of $V_i(t)$ in any time interval $t \in [t_0, T]$, $T \ge h$, between switches.

B. Effect of the sign change

In the previous section we have proven that in between sign changes each of the Lyapunov functions V_i decreases. However, given the differences between each P_i, the level sets of the Lyapunov functions do not necessarily contain each other. Thus, after a switch, we can not initialize the next Lyapunov function with the same value as the previous one. To address this issue, we have to analyse when $c_i \ge V_i$ implies that $c_j \ge V_j$ for some $c_i, c_j > 0$, $i \ne j$, $i, j = \{1,2,3,4\}$. This can be done using the S-Procedure [31, pp. 655] (see also [19, Lemma 1]). Following the S-Procedure, we need to find $\eta > 0$ such that

$$\eta\begin{bmatrix} P_i & 0 \\ 0 & -c_i \end{bmatrix} - \begin{bmatrix} P_j & 0 \\ 0 & -c_j \end{bmatrix} \ge 0.$$

This automatically means that $c_j \ge \eta\,c_i$. Given the structure of each P_i, η has to be greater than 1. Now, we can focus our analysis in the first block of the resulting matrix:

$$\begin{bmatrix} (\eta-1)p_1\mathbf{I}_2 & \eta\mathbf{p}_i - \mathbf{p}_j \\ \star & (\eta-1)p_2 \end{bmatrix}.$$

To find conditions over η that renders this matrix positive semidefinite, we can use again the Schur complement, which in this case is:

$$(\eta-1)p_1\mathbf{I}_2 - \frac{1}{(\eta-1)p_2}\big(\eta\,\mathbf{p}_i - \mathbf{p}_j\big)\big(\eta\,\mathbf{p}_i - \mathbf{p}_j\big)^\top \ge \\ (\eta-1)p_1\mathbf{I}_2 - \frac{\|\eta\mathbf{p}_i - \mathbf{p}_j\|^2}{(\eta-1)p_2}\mathbf{I}_2 \ge 0.$$

Additionally, from Assumption 1.2 and the structure of $y(t)$, we know in which order we have to switch the Lyapunov functions: $V_1 \to V_2 \to V_3 \to V_4 \to V_1$. Then, we only need to compute the difference between contiguous $\mathbf{p}_i$, which yields in all cases $\|\eta\mathbf{p}_i - \mathbf{p}_j\|^2 = 2(1+\eta^2)$. Using these bounds, we found the following condition that ensure the level set inclusion:

$$\eta \ge \frac{p_1p_2 + 2\sqrt{p_1p_2-1}}{p_1p_2 - 2} = 1 + \frac{2+2\sqrt{p_1p_2-1}}{p_1p_2-1}. \tag{20}$$

Note that when $p_1p_2 \to \infty$, $\eta \to 1$. Then, η is always greater than one.

If we take the equality in (20), the level sets are not proper contained, but rather they are tangent. Taking this value for η, we can initialize the Lyapunov function as $V_j = \eta\,V_i$, where j follows i in the order discussed above.

After the n-th switch, the Lyapunov function in turn can be bounded in terms of the initial value of the first Lyapunov function as:

$$V_i(t_0 + n\,h) \le \left(\eta^{\frac{n-1}{2}}V_1^{\frac{1}{2}}(t_0) - \frac{\alpha\,h}{2}\sum_{i=1}^{n}\eta^{\frac{i-1}{2}}\right)^2.$$

Then, to have convergence, the sum term has to *grow* faster than $\eta^{\frac{n-1}{2}}$. Consider the term inside the brackets. If we divide it by the square root of the initial condition and we simplify the sum, we get $\eta^{\frac{n-1}{2}} - c(\eta^{\frac{n}{2}}-1)/(\eta^{\frac{1}{2}}-1)$, where $c = \alpha\,h/(2\,V_1^{\frac{1}{2}}(t_0))$. Then, if there exists an integer $n > 0$ such that $\eta^{\frac{n-1}{2}}\big(1-(1-c)\eta^{\frac{1}{2}}\big) \ge c$, there is finite-time convergence. This happens iff $c \ge 1 - 1/\eta^{\frac{1}{2}}$ or, equivalently,

$$\frac{1}{4}\gamma\,h\,\lambda_{\min}^{\frac{1}{2}}(P_i) \ge V_1^{\frac{1}{2}}(t_0)\left(1 - \frac{1}{\eta^{\frac{1}{2}}}\right). \tag{21}$$

Notice that this conditions tie the convergence to the initial error. However, for any positive γ and h, the uniform finite-time convergence can be ensured for a sufficiently small initial error.

C. Convergence conditions

To guarantee the uniform finite-time stability of $\{e(t) = 0, x(t) = 0\}$ we have to chose positive p_2, γ, k_1 and k_2 that satisfy (17). Then, using these values we have to compute η as in (20) together with the smallest eigenvalue of each P_i and check if (21) is met for a given h and $V_i(t_0)$. Up to this point, we have as degree of freedom the constants mentioned above. With the purpose of giving a precise set of gains, we start by setting $p_2 = 1$ and $\gamma = A/2$. This simplifies (17) and by the introduction of $c > 0$ allows to choose k_1 and k_2 as in (8). With the use of these gains, one has to verify (21) which reduces to

$$\frac{1}{8}A\,h\,\lambda_{\min}^{\frac{1}{2}}(P_i) \ge V_1^{\frac{1}{2}}(t_0)\left(1 - \frac{1}{\eta^{\frac{1}{2}}}\right), \tag{22}$$

where η can be computed as in (9). Note that for all $i = \{1,2,3,4\}$ we have

$$\lambda_{\min}(P_i) = \frac{1}{2}\Big(p_1 + p_2 - \big(8 + p_1^2 - 2\,p_1p_2 + p_2^2\big)^{\frac{1}{2}}\Big), \\ \lambda_{\max}(P_i) = \frac{1}{2}\Big(p_1 + p_2 + \big(8 + p_1^2 - 2\,p_1p_2 + p_2^2\big)^{\frac{1}{2}}\Big),$$

and by substituting $p_1 = 2\,k_2 - k_1/A$ and $p_2 = 1$, the above expressions are equivalent to λ_- and λ_+ defined in (10). Since $\lambda_{\max}(P_i)\|\zeta(t_0)\|^2 \ge V_i(t_0)$, then $A\,h\,\lambda_{\min}^{1/2}(P_i)/8 \ge \lambda_{\max}^{1/2}(P_i)\|\zeta(t_0)\|(1 - 1/\eta^{1/2})$ implies that (22) is satisfied. This condition is the one used in Theorem 1, which hence

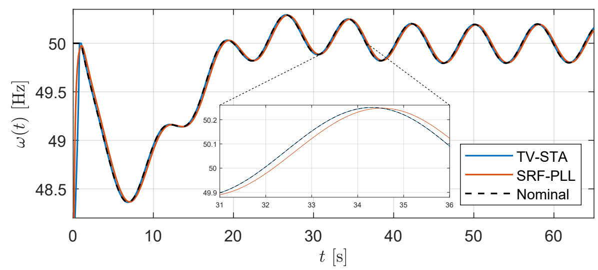

Fig. 1. Frequency tracking exhibited by the proposed estimator (7) and the SRF-PLL.

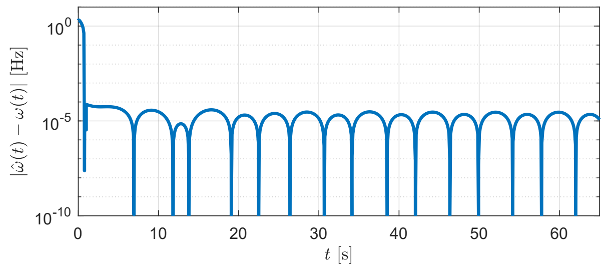

Fig. 2. Logarithmic plot of the frequency estimation error with the proposed algorithm (7).

summarizes all findings of the derivations detailed in this section.

VI. Simulation Example

We test the proposed frequency estimator (7) by using an AC signal of unitary amplitude, i.e., $A = 1$. We assume that initially its frequency is constant for $t \in [0, 1)$, i.e., $\omega(t) = 2\,\pi\,50$. This corresponds to a nominal, stationary power system operation. Then, at $t = 1$ [s] a disturbance occurs and, as a consequence, the frequency starts to vary. We assume that this variation can be described by the following signal $\omega(t) = 2\,\pi(50 - 4\exp(-0.13(t-1))\sin(0.15(t-1)) + 0.2\sin(0.8(t-1)))$ for $t \geq 1$. This frequency signal follows the under-frequency profile proposed in [32] plus an oscillating term. A direct calculation shows that the derivative of $\omega(t)$ is bounded by $\Delta = 3 \geq |\dot{\omega}(t)|$.

For the given signal amplitude $A = 1$, $\Delta = 3$, and for $c = 16.05$, we obtain the gains $k_1 = 17.7$ and $k_2 = 50$ from Theorem 1 for the proposed estimator (7). Its initial conditions are set to $\hat{y}_1(0) = 1$, $\hat{y}_2(0) = 0$, and $\hat{\omega}(0) = 2\,\pi\,48$.

We compare the performance of our proposed approach with the standard SRF-PLL [4]. The PI filter of the SRF-PLL is configured with a proportional gain of $k_p = 13 \times 10^3$ and an integral gain of $k_i = 60 \times 10^3$. The result of the estimation process is shown in Figure 1. It can be seen that the proposed time-varying super-twisting algorithm (TV-STA) tracks the frequency exactly, whereas the SRF-PLL introduces a “phase shift”. In Figure 2 the absolute value of the frequency estimation error for the TV-STA is shown on a logarithmic scale. The high slope at the beginning is an indicator of the finite-time convergence. In Figure 3 the absolute values of the errors produced by the different estimators are presented. Whereas the TV-STA provides the exact value, the SRF-PLL commits an error as high as 60 [mHz].

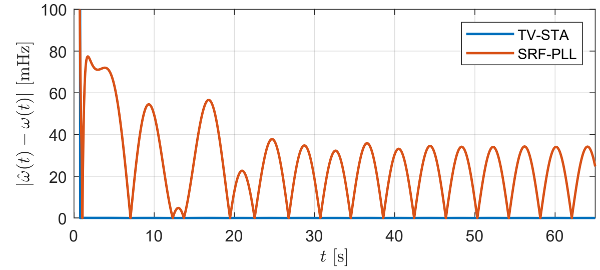

Fig. 3. Absolute value of the frequency tracking error corresponding to the proposed algorithm (7) and the SRF-PLL.

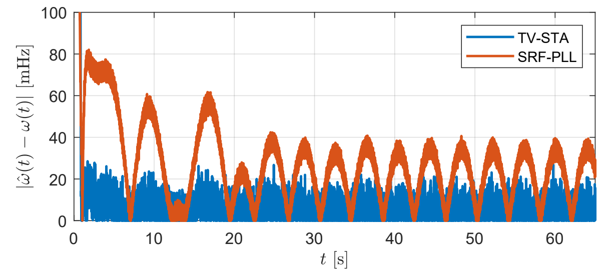

Fig. 4. Absolute value of the frequency tracking error in the presence of noise corresponding to the proposed algorithm (7) and the SRF-PLL.

The algorithms are also tested in the presence of noise. For this purpose, a Gaussian noise of zero mean and 7×10^{-6} variance was added to the measured signal (around 1% of the signal amplitude). The estimation results are shown in Figure 4. It can be seen that the performance of the TV-STA is more affected by the noise than that of the SRF-PLL. However, the effect of the time variation of the frequency still has a very significant impact on the SRF-PLL performance. In Figure 5, it can also be seen that despite the presence of noise the proposed TV-STA does not introduce a “phase shift” in the estimate, while the SRF-PLL does, hence making the TV-STA more competitive even in this noisy scenario. We note that the integral gain k_i of the SRF-PLL was chosen fairly large to compensate for the time variation of the frequency. However, enlarging it even more will mainly increase the effect of the noise rather than mitigate the effect of the frequency variation.

VII. Conclusions

In this work, a time-varying version of the super twisting algorithm is proposed to estimate the instantaneous frequency of a symmetric three phase AC signal. In contrast to existing approaches, the algorithm is capable of exactly tracking a time-varying frequency signal. An expression for the algorithm’s gains is provided in terms of the signal

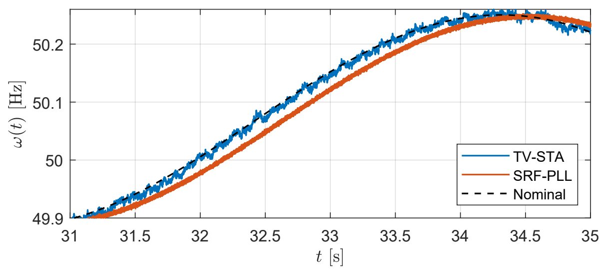

Fig. 5. Zoomed-in snapshot of the frequency tracking in the presence of noise exhibited by the proposed estimator (7) and the SRF-PLL.

amplitude and its maximum rate of change. These gains ensure the algorithm's convergence when the initial error is sufficiently small, i.e., it is a local convergence result. However, it is the authors believe that this restriction is imposed by the method of proof and that it is not an inherent property of the algorithm.

The performance of the algorithm is illustrated in simulation examples under nominal and disturbed circumstances. This performance is compared with a standard SRF-PLL. As established in the analysis, the proposed method is capable of exactly tracking a time-varying frequency under nominal conditions, whereas the SRF-PLL exhibits a significant error due to the time variation of the frequency. In the presence of noise, the exact convergence property is lost, but the estimate can closely track the frequency profile without dephasing, contrary to what happens with the SRF-PLL.

Future work aims to establish a global result, a broader set of admissible gains and to extend the applicability of the proposed algorithm to the case of measurements corrupted by harmonics.

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