A consensus-based voltage control for reactive power sharing and PCC voltage regulation in microgrids with parallel-connected inverters*

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***Abstract*— We consider small-scale power systems consisting of several inverter-interfaced units connected in parallel to a common bus, the point of common coupling (PCC), and sharing a joint load. This is a frequently encountered configuration in microgrid applications. In such a setting, two important control objectives are reactive power sharing and voltage regulation at the PCC. In this paper, we first show that the nonlinear equilibrium equations corresponding to the aforementioned objectives admit a unique positive solution. Then, we propose a consensus-based distributed voltage controller which renders this desired unique solution locally asymptotically stable. Finally, control performance is illustrated via a simulation example.**

I. Introduction

Worldwide the amount of renewable energy sources (RESs) in power grids is increasing day by day. In the process of RES integration, power electronic inverters play a major role since they represent the main interfaces between the RESs and the grid [1]. In such a setup with a large number of RESs, to improve system redundancy and reliability, inverters are typically connected in parallel to a common bus called the point of common coupling (PCC) [2]. In this paper, such a network is termed a *parallel* microgrid (MG). Some examples of parallel MGs are battery power plants (also known as battery energy storage systems) and distributed UPS (uninterruptible power supply) systems [1], [3], [4]. Similar to any standalone AC power network, there are various challenging control objectives to be addressed in a parallel MG [2], [5].

As mentioned in [1]–[12], maintaining all the bus voltage magnitudes within certain limits is an important control objective in parallel MGs. In such networks, the most critical voltage magnitude is at the PCC since there is no generation unit present at that node [2], [6]. On the other hand, it is also of great relevance to share the reactive power demand of the system loads proportionally between the distributed generators (DGs) [2]–[11]. The latter objective is of particular relevance in MGs, since in such networks, the power lines are usually rather short and, thus, small differences in voltage amplitudes can lead to high reactive power flows between DGs [13]–[15].

The problem of power sharing in parallel MGs has been investigated in [2]–[4], [6]–[11] using decentralized voltage droop or droop-like approaches. A distributed control law has been proposed in [5] for addressing the same problem. In [2], [6], the objective of PCC voltage regulation has also been included. However, the aforementioned approaches are mainly simulation and/or experiment-based studies and assume that there exists a stable equilibrium point. In contrast to that, [12] has proposed a quadratic voltage droop controller to ensure voltage stability in lossless MGs. However, accurate proportional reactive power sharing cannot be guaranteed without risking system stability.

In light of the above-mentioned limitations, our main contributions in this paper are as follows:

1) We show that in the case of a lossless parallel MG with a constant current load connected at the PCC, there exists a unique stationary solution satisfying accurate reactive power sharing and PCC voltage regulation.
2) Inspired by consensus algorithms, see for e.g. [16], we propose a distributed voltage controller, such that the stationary solutions of the closed-loop system satisfy the aforementioned control objectives.
3) We derive a sufficient condition for local asymptotic stability of this unique closed-loop equilibrium point.

In contrast to [2]–[11], we provide a rigorous mathematical analysis corroborating the aforementioned properties. The proposed approach can be implemented locally at each DG and can achieve steady-state reactive power sharing by using local reactive power injections and reactive power injections of neighboring units communicated over a sparse network. Furthermore, the objective of PCC voltage regulation can be achieved without the PCC voltage being communicated to all DGs, which significantly reduces the communication effort compared to a centralized solution.

The remainder of the paper is organized as follows. In Section II, we introduce the MG model and formalize the control objectives. In Section III, we show that there exists a unique stationary solution satisfying the two control objectives focused on in this paper. In Section IV, we propose a distributed voltage controller to achieve these control objectives. Furthermore, we provide a sufficient condition for local asymptotic stability. In Section V, robustness of the presented approach in the presence of load variations and model uncertainties is evaluated. Finally, in Section VI, we draw some conclusions and point out future research directions.

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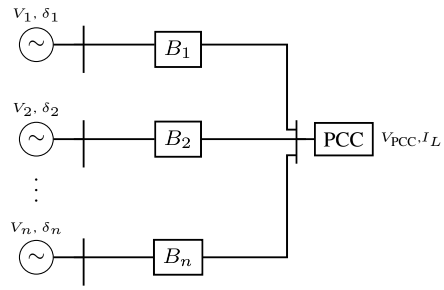

Fig. 1: Schematic representation of a parallel MG.

Notation. We denote by I_n the $n \times n$ identity matrix, by $\mathbb{0}_{n\times m}$ the $n \times m$ matrix with all entries equal to zero, by $\mathbb{1}_n$ the vector with all entries being equal to one and by $\mathbb{0}_n$ the zero vector. Let $Z \in \mathbb{R}^{n\times n}$ be a symmetric matrix. If Z is positive (negative) definite, we denote this by $Z > 0$ ($Z < 0$). If Z is positive (negative) semidefinite, we denote this by $Z \geq 0$ ($Z \leq 0$). The elements under the diagonal of Z is denoted by $*$. Moreover, $\ker(Z)$ denotes the kernel of Z and $\text{trace}(Z)$ its trace. Let $x = \text{col}(x_i)$ denote a column vector with entries x_i. Then, $[x]$ denotes the diagonal matrix with diagonal entries x_i. Finally, for a function $f : \mathbb{R}^n \to \mathbb{R}$, ∇f denotes its gradient.

II. Microgrid Model and Control Objectives

A. Microgrid Model

We consider a MG with $n > 1$ inverter-interfaced DGs, which are all connected in parallel to the PCC, see Figure 1. We denote the set of units connected to the PCC by $\mathcal{N} = \{1, 2, \dots, n\}$. Furthermore, we assume that the power lines are lossless, i.e., all lines can be represented by pure susceptances. This can be justified as follows. In medium voltage (MV) grids, the output impedance of a DG is typically inductive due to the presence of an output inductor and/or an output transformer. In that case the impedance of the power line together with the DG output impedance is dominated by the inductive part [3], [6], [14]. We only consider such MGs in this paper and denote the susceptance connecting the PCC and the i-th unit by $B_i \in \mathbb{R}_{<0}$, see Figure 1.

We assign to each DG a phase angle $\delta_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and a voltage magnitude $V_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$. The vector of voltage magnitudes is denoted by $V = \text{col}(V_i) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n_{>0}$. Furthermore, we assume that all phase angles are expressed relative to the phase angle of the three-phase voltage at the PCC and denote the voltage magnitude at the PCC by $V_{\text{PCC}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$. Then, the active and reactive power flows from the i-th DG to the PCC are given by [17, Section 4.4]

$$\begin{aligned} P_i\left(\delta_i(t), V_i(t), V_{\text{PCC}}(t)\right) &= |B_i| V_i(t) V_{\text{PCC}}(t) \sin(\delta_i(t)), \\ Q_i\left(\delta_i(t), V_i(t), V_{\text{PCC}}(t)\right) &= |B_i| V_i^2(t) \\ &\quad - |B_i| V_i(t) V_{\text{PCC}}(t) \cos(\delta_i(t)). \end{aligned} \tag{II.1}$$

In the sequel, unless confusion arises, we will not explicitly display the dependence of voltages and phase angles on time.

For our analysis, we make the following standard decoupling assumption [18].

Assumption 2.1: $\delta_i(t) < \epsilon \quad \forall t \geq 0, \quad i \in \mathcal{N}$, where $\epsilon \in \mathbb{R}$ with $|\epsilon| \ll 1$.

Under Assumption 2.1, $\cos(\delta_i) \approx 1$ and thus the reactive power flow Q_i in (II.1) becomes independent of δ_i, i.e.,

$$Q_i(V_i, V_{\text{PCC}}) = |B_i| V_i \left(V_i - V_{\text{PCC}}\right). \tag{II.2}$$

By defining

$$B = \text{diag}(|B_i|) \in \mathbb{R}^{n\times n}, \tag{II.3}$$

the vector of DG-side reactive power flows $Q_I \in \mathbb{R}^n$ can be compactly written as

$$Q_I(V, V_{\text{PCC}}) = \text{col}(Q_i) = [V]BV - V_{\text{PCC}}BV. \tag{II.4}$$

Since we are mainly interested in voltage aspects, we neglect the frequency dynamics and represent the DGs by the standard model [14], [19]

$$V = V^d + u^V, \tag{II.5}$$

where $V^d \in \mathbb{R}^n_{>0}$ is the desired DG voltage amplitude and $u^V : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a control signal. Moreover, we assume that the loads in the MG are represented by a single equivalent constant current load $I_L \in \mathbb{R}_{<0}$ connected at the PCC[1], see Figure 1. By Kirchoff's current law, the current balance at the PCC is given by

$$I_L = \sum_{i=1}^{n} |B_i|(V_{\text{PCC}} - V_i) = V_{\text{PCC}} \mathbb{1}_n^\top B \mathbb{1}_n - \mathbb{1}_n^\top BV. \tag{II.6}$$

Note that the PCC voltage V_{PCC} implicitly depends on the bus voltages V_i. By using the current balance (II.6), this relation can be made explicit, i.e.,

$$V_{\text{PCC}}(V) = \frac{I_L + \mathbb{1}_n^\top BV}{\mathbb{1}_n^\top B \mathbb{1}_n}. \tag{II.7}$$

This fact is used in our subsequent analysis.

B. Problem Statement

To state the considered problem in this paper, we introduce the following quantities. Let the desired voltage magnitude at the PCC be denoted by $V^{\text{d}}_{\text{PCC}} \in \mathbb{R}_{>0}$, let $A = \text{diag}(a_i) \in \mathbb{R}^{n\times n}_{>0}$ denote a diagonal weighting matrix and let $\mathcal{L} \in \mathbb{R}^{n\times n}$ denote the Laplacian matrix of an undirected and connected graph. Then, $\mathcal{L} \geq 0$ with $\mathcal{L}\mathbb{1}_n = \mathbb{0}_n$ [20]. We seek to solve the following control problem.

Problem 2.2: Consider the system (II.4), (II.5), (II.7). Design a control law for the control input u^V, such that the solutions of the system (II.4), (II.5), (II.7) converge asymptotically to a stationary voltage solution $V^* \in \mathbb{R}^n_{>0}$ with the following two properties:

1) Voltage regulation at the PCC, i.e.,

$$V_{\text{PCC}}(V^*) = V^d_{\text{PCC}} \quad \text{and} \tag{II.8}$$

[1] In this paper, we employ generator convention [18]. Thus, $I_L < 0$ means that the load is inductive. In practice, most of the loads have an inductive behavior [18, Chapter 7] and hence the assumption $I_L < 0$ is realistic.

2) Reactive power sharing among DGs, i.e.,

$$AQ_I(V^*, V_{\text{PCC}}(V^*)) = \alpha \mathbb{1}_n, \text{ for some } \alpha \in \mathbb{R}. \tag{II.9}$$

We remark that, compared to the related work [14], [15], [21], the consideration of a parallel MG topology results in a clear voltage regulation objective, i.e., 1) in Problem 2.2. In addition, as we show in the sequel, this type of voltage regulation objective does not conflict with that of reactive power sharing.

III. Existence of a unique stationary solution

We begin our analysis by investigating existence and uniqueness properties of stationary solutions to the network equations (II.4), (II.5), (II.7) under the requirements of Problem 2.2. As can be readily verified, this is equivalent to investigating solutions to the set of equations (II.8), (II.9). Note that (II.8), (II.9) is a system of $n+1$ equations in n unknowns $V_i^*, i \in \mathcal{N}, |\mathcal{N}| = n$ and hence is an overdetermined system of nonlinear equations. Explicitly solving such an overdetermined system of nonlinear equations can be difficult. Yet, the following result shows that for any $A > 0$ and $I_L < 0$ there is exactly one $V^* \in \mathbb{R}^n_{>0}$, which satisfies (II.8) and (II.9) simultaneously.

Lemma 3.1: For any given $V^d_{\text{PCC}} > 0$, $A > 0$, and $I_L < 0$, there exists a unique vector $V^* \in \mathbb{R}^n_{>0}$ that satisfies both (II.8) and (II.9).

Proof: Consider (II.8) and (II.9). Note that any $V^* \in \mathbb{R}^n_{>0}$ satisfying (II.9) and (II.8) simultaneously has to satisfy

$$AQ_I(V^*, V^d_{\text{PCC}}) = \alpha \mathbb{1}_n, \tag{III.1}$$

which by (II.2) can be written as

$$a_i |B_i| V_i^* \left(V_i^* - V^d_{\text{PCC}}\right) = \alpha, \quad i \in \mathcal{N}. \tag{III.2}$$

Furthermore, left-multiplying (III.1) with $\mathbb{1}_n^\top A^{-1}[V^*]^{-1}$ yields

$$\mathbb{1}_n^\top [V^*]^{-1} Q_I(V^*, V^d_{\text{PCC}}) = -I_L = \alpha \mathbb{1}_n^\top [V^*]^{-1} A^{-1} \mathbb{1}_n, \tag{III.3}$$

where we have used (II.4) and (II.6) to obtain the first equality. From (III.3) and the fact that $I_L < 0$, it is easy to see that necessarily $\alpha > 0$, which means that (III.2) has one negative and one positive solution for V_i^*. Denoting the latter by V_i^+, we have

$$V_i^+ = \frac{a_i|B_i|V^d_{\text{PCC}} + \sqrt{(a_i|B_i|V^d_{\text{PCC}})^2 + 4a_i|B_i|\alpha}}{2a_i|B_i|} := f_i(\alpha). \tag{III.4}$$

Now, bearing in mind (III.4), it remains to show that there exists a unique $\alpha > 0$ that satisfies (III.3), which in terms of α can be written as

$$|I_L| = \alpha \sum_{i \in \mathcal{N}} \frac{1}{a_i f_i(\alpha)} := g(\alpha). \tag{III.5}$$

Note that, if such α exists and is unique, then $V^* = \text{col}(V_i^+) \in \mathbb{R}^n_{>0}$ is the unique solution to the algebraic equations (II.8)-(II.9).

From (III.5), clearly we have

$$\lim_{\alpha \to 0} g(\alpha) = 0, \qquad \lim_{\alpha \to +\infty} g(\alpha) = +\infty.$$

Hence, by continuity of g, there exists $\alpha > 0$ that satisfies (III.5). To prove uniqueness, we show that g is a strictly increasing function. For this purpose, differentiating $g(\alpha)$ with respect to α gives

$$g'(\alpha) = \sum_{i \in \mathcal{N}} \frac{1}{a_i f_i(\alpha)} - \alpha \sum_{i \in \mathcal{N}} \frac{f_i'(\alpha)}{a_i f_i^2(\alpha)} = \sum_{i \in \mathcal{N}} \frac{f_i(\alpha) - \alpha f_i'(\alpha)}{a_i f_i^2(\alpha)}.$$

Hence, g is strictly increasing if

$$f_i(\alpha) - \alpha f_i'(\alpha) > 0, \tag{III.6}$$

for each $i \in \mathcal{N}$. Since $f_i(\alpha)$ given by (III.4) is concave in $\mathbb{R}_{\geq 0}$, we have

$$f_i(0) \leq f_i(\alpha) + f'(\alpha)(0 - \alpha).$$

Noting that $f_i(0) = V^d_{\text{PCC}} > 0$, the latter inequality implies (III.6), hence confirming the uniqueness of $\alpha > 0$ which satisfies (III.5). Therefore, as mentioned earlier, corresponding to this unique $\alpha > 0$, there exists a unique $V_i^+ = f_i(\alpha) \in \mathbb{R}_{>0}$, see (III.4). This yields that there exists a unique vector $V^* = \text{col}(V_i^+) \in \mathbb{R}^n_{>0}$ satisfying (II.8)-(II.9), completing the proof. ■

IV. A Distributed Control law for Reactive Power Sharing and PCC Voltage Regulation

In this section, we propose a distributed voltage control law to address Problem 2.2. Consider the model (II.5). We design the control input u^V such that

$$\dot{u}^V = -\kappa[V] A \mathcal{L} A Q_I(V) - \tilde{V}_{\text{PCC}}(V) E \mathbb{1}_n, \tag{IV.1}$$

where $\kappa \in \mathbb{R}_{>0}$ is a controller parameter, $\mathcal{L} \in \mathbb{R}^{n \times n}$ is the Laplacian matrix of a connected undirected graph, $E \in \mathbb{R}^{n \times n}$ is a diagonal *pinning gain matrix* which has positive entries only for the units which have access to the measurement $\tilde{V}_{\text{PCC}} = V_{\text{PCC}} - V^d_{\text{PCC}}$. This implies that the quantity $\tilde{V}_{\text{PCC}}$ is not required at all DG units, thereby significantly relaxing the communication requirements. Obviously, we assume that the matrix E has at least one nonzero element, i.e., $E \geq 0$. Furthermore, $[V] = \text{diag}(V_i) \in \mathbb{R}^{n \times n}_{>0}$ and Q_I is defined in (II.4).

By combining (II.5) and (IV.1) and recalling that $V^d \in \mathbb{R}^n_{>0}$, we obtain the following closed-loop dynamics:

$$\dot{V} = -\kappa[V] A \mathcal{L} A Q_I(V) - \tilde{V}_{\text{PCC}}(V) E \mathbb{1}_n, \quad V(0) = V^d. \tag{IV.2}$$

A. Preliminary Lemmata

The following lemma shows that (IV.2) has a unique equilibrium point, which is exactly the unique voltage vector given in the proof of Lemma 3.1 and therefore simultaneously satisfies voltage regulation and reactive power sharing objectives.

Lemma 4.1: The vector $V^* \in \mathbb{R}^n_{>0}$ in Lemma 3.1 is the unique equilibrium point of (IV.2).

Proof: In the proof of Lemma 3.1, we have shown that V^* satisfies (II.9) and (II.8). Hence, $\tilde{V}_{\text{PCC}}(V^*) = 0$ and

$\mathcal{L}AQ_I(V^*) = \alpha\mathcal{L}\mathbb{1}_n = \mathbb{0}_n$, where the latter equality is an immediate consequence of the properties of the Laplacian. Therefore, V^* is an equilibrium point of (IV.2).

Next, we show by contradiction that V^* is the only equilibrium point of the system (IV.2). Suppose that for the system (IV.2), there exists another equilibrium solution $V^s \in \mathbb{R}^n_{>0}$, i.e.,

$$\mathbb{0}_n = \kappa[V^s]A\mathcal{L}AQ_I(V^s) + \tilde{V}_{\text{PCC}}(V^s)E\mathbb{1}_n. \tag{IV.3}$$

Since $[V^s] > 0$, left-multiplying (IV.3) with $\mathbb{1}_n^\top A^{-1}[V^s]^{-1}$ yields

$$\tilde{V}_{\text{PCC}}(V^s)\text{trace}(A^{-1}[V^s]^{-1}E) = 0.$$

Due to the fact that $\text{trace}(A^{-1}[V^s]^{-1}E) > 0$, the above equality is satisfied if and only if $\tilde{V}_{\text{PCC}}(V^s) = 0$. But this implies that V^s also has to satisfy

$$\mathcal{L}AQ_I(V^s) = \mathbb{0}_n$$

which, since $\mathcal{L}$ is the Laplacian matrix of an undirected, connected graph, is equivalent to

$$AQ_I(V^s) = \alpha\mathbb{1}_n, \alpha \in \mathbb{R}.$$

Hence, V^s must satisfy both (II.9) and (II.8). But Lemma 3.1 specifies that V^* is the only solution to (II.9) and (II.8), which implies $V^s = V^*$. This completes the proof. ■

Let $J(V) \in \mathbb{R}^{n\times n}$ denote the partial derivative of Q_I (given by (II.4), (II.6)) with respect to V, i.e.,

$$\begin{aligned} J(V) &= \nabla Q_I \\ &= 2B[V] - \frac{1}{\text{trace}(B)}B[V]\mathbb{1}_n\mathbb{1}_n^\top B - V_{\text{PCC}}(V)B. \end{aligned} \tag{IV.4}$$

We establish the following property of $J(V)$.

Lemma 4.2: The matrix $J(V^*)$ with $J(V)$ defined in (IV.4) is invertible.

Proof: Recall that $V_{\text{PCC}}(V^*) = V^d_{\text{PCC}}$, see (II.8). Hence,

$$\begin{aligned} J(V^*) &= 2B[V^*] - \frac{1}{\text{trace}(B)}B[V^*]\mathbb{1}_n\mathbb{1}_n^\top B - V^d_{\text{PCC}}B, \\ &= B\left([V^*] - V^d_{\text{PCC}}I_n\right) \\ &\quad + \frac{1}{\text{trace}(B)}[V^*]\left(\text{trace}(B)B - B\mathbb{1}_n\mathbb{1}_n^\top B\right). \end{aligned} \tag{IV.5}$$

Since $[V^*]$ is invertible, a necessary and sufficient condition for $J(V^*)$ to have the same property is that the matrix product $S(V^*) := J(V^*)[V^*]$ has full rank. To investigate the properties of $S(V^*)$, we note that $S(V^*) = S(V^*)^\top$ and write it as the following matrix sum

$$S(V^*) = S_1(V^*) + S_2(V^*), \tag{IV.6}$$

where

$$\begin{aligned} S_1(V^*) &= B\left([V^*] - V^d_{\text{PCC}}I_n\right)[V^*], \\ S_2(V^*) &= \frac{1}{\text{trace}(B)}[V^*]\left(\text{trace}(B)B - B\mathbb{1}_n\mathbb{1}_n^\top B\right)[V^*]. \end{aligned} \tag{IV.7}$$

The diagonal matrix $S_1(V^*)$ given in (IV.7) can be equivalently expressed as

$$S_1(V^*) = B[V^*]^2 - V^d_{\text{PCC}}B[V^*] = [Q_I(V^*, V^d_{\text{PCC}})] = \alpha A^{-1},$$

where we have used (II.4) and (III.1) to write the second and third equalities above, respectively. At $V^* \in \mathbb{R}^n_{>0}$, from Lemma 3.1, we have that $\alpha > 0$. In addition, $A > 0$. Therefore, $S_1(V^*) > 0$.

Next, consider the symmetric matrix $S_2(V^*)$ given in (IV.7). Since $B > 0$ is a diagonal matrix, $\text{trace}(B)B - B\mathbb{1}_n\mathbb{1}_n^\top B$ has positive diagonal entries and has the property that

$$\left(\text{trace}(B)B - B\mathbb{1}_n\mathbb{1}_n^\top B\right)\mathbb{1}_n = \mathbb{0}_n.$$

Hence, by Gershgorin's theorem [22],

$$\text{trace}(B)B - B\mathbb{1}_n\mathbb{1}_n^\top B \geq 0.$$

By noting that $\text{trace}(B) \in \mathbb{R}_{>0}$ and $[V^*] > 0$, we have that $S_2(V^*) \geq 0$. Consequently, $S(V^*) = S_1(V^*) + S_2(V^*) > 0$, completing the proof. ■

B. A Condition for Asymptotic Stability

We are now in the position to formulate our main result, which provides a solution to Problem 2.2.

Proposition 4.3: The unique equilibrium point $V^* \in \mathbb{R}^n_{>0}$ of the system (IV.2) is locally asymptotically stable if the controller gains κ and E are chosen such that

$$\Phi(V^*) := \begin{bmatrix} \kappa J(V^*)[V^*] & \frac{1}{2}\left(J(V^*)E\mathbb{1}_n + \kappa[V^*]B\mathbb{1}_n\right) \\ * & \text{trace}(BE) \end{bmatrix} > 0. \tag{IV.8}$$

Proof: Consider the Lyapunov function candidate

$$\mathcal{F}(V) = \frac{1}{2}Q_I^\top A\mathcal{L}AQ_I + \frac{1}{2}(\tilde{V}_{\text{PCC}})^2\text{trace}(B), \tag{IV.9}$$

where we recall that $\tilde{V}_{\text{PCC}} = V_{\text{PCC}} - V^d_{\text{PCC}}$ with V_{PCC} and Q_I given by (II.7) and (II.4) respectively.

At $V = V^*$, we have that $Q_I(V^*) = \alpha A^{-1}\mathbb{1}_n$ and $\tilde{V}_{\text{PCC}}(V^*) = V_{\text{PCC}}(V^*) - V^d_{\text{PCC}} = 0$, see (II.9) and (II.8). Hence, $\mathcal{F}(V^*) = 0$. Furthermore, the gradient of $\mathcal{F}$ with respect to V is given by

$$\nabla\mathcal{F} = J^\top A\mathcal{L}AQ_I + \tilde{V}_{\text{PCC}}B\mathbb{1}_n, \tag{IV.10}$$

where J is defined in (IV.4). Evaluating the gradient at $V = V^*$, we find that $\nabla\mathcal{F}|_{V^*} = \mathbb{0}_n$. Therefore, V^* is a critical point of $\mathcal{F}$ [23], [24].

The Hessian of $\mathcal{F}$ is given by

$$\nabla^2\mathcal{F} = J^\top A\mathcal{L}AJ + [A\mathcal{L}AQ_I]^\top\frac{\partial J}{\partial V} + \frac{1}{\text{trace}(B)}B\mathbb{1}_n\mathbb{1}_n^\top B. \tag{IV.11}$$

By recalling that at $V = V^*$, $AQ_I(V^*) = \alpha\mathbb{1}_n$, $\alpha \in \mathbb{R}$ we obtain from (IV.11) that

$$\nabla^2\mathcal{F}|_{V^*} = J(V^*)^\top A\mathcal{L}AJ(V^*) + \frac{1}{\text{trace}(B)}B\mathbb{1}_n\mathbb{1}_n^\top B. \tag{IV.12}$$

We next show that $\nabla^2\mathcal{F}|_{V^*} > 0$. By Lemma 4.2, $J(V^*)$ is invertible. Furthermore, $A > 0$. Hence, the matrix $J(V^*)^\top A\mathcal{L}AJ(V^*)$ is similar to the Laplacian matrix $\mathcal{L}$, which - under the standing assumptions - is positive semidefinite with a simple zero eigenvalue and a corresponding right-eigenvector $\mathbb{1}_n$. Consequently, $J(V^*)^\top A\mathcal{L}AJ(V^*)$ is positive semidefinite with a simple zero eigenvalue and a corresponding right-eigenvector $(AJ(V^*))^{-1}\mathbb{1}_n$. Likewise, since $B > 0$ the matrix $B\mathbb{1}_n\mathbb{1}_n^\top B$ is positive semidefinite with a zero eigenvalue of algebraic multiplicity $n-1$ with corresponding right-eigenvectors $B^{-1}w_i$, where

$$\begin{aligned} w_1 &= \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \end{bmatrix}^\top, \\ &\vdots \\ w_{n-1} &= \begin{bmatrix} 0 & \dots & 0 & 1 & -1 \end{bmatrix}^\top, \quad i = 1, \dots, n-1. \end{aligned} \tag{IV.13}$$

Hence, it remains to show that the kernels of the two matrices in the matrix sum (IV.12) don't intersect. To show this, it suffices to show that for any w_i,

$$AJ(V^*)B^{-1}w_i \neq \beta\mathbb{1}_n, \quad \forall\beta \in \mathbb{R}. \tag{IV.14}$$

From (IV.5), we obtain

$$J(V^*)B^{-1}w_i = \left(2[V^*] - V_{\text{PCC}}^d I_n\right) w_i.$$

It follows from (III.4) that $V_i^* > V_{\text{PCC}}^d$, $i = 1, \dots, n$. Hence, the matrix $\left(2[V^*] - V_{\text{PCC}}^d I_n\right)$ is a diagonal matrix with positive diagonal entries. Since the same applies to A, no w_i in (IV.13) can satisfy (IV.14). Consequently, $\nabla^2\mathcal{F}|_{V^*} > 0$ and V^* is a local minimum of the function $\mathcal{F}$.

Calculating the time derivative of $\mathcal{F}$ along the dynamics (IV.2) yields

$$\begin{aligned} \dot{\mathcal{F}} &= \nabla\mathcal{F}^\top \dot{V}, \\ &= -\kappa Q_I^\top A\mathcal{L}AJ[V]A\mathcal{L}AQ_I - Q_I^\top A\mathcal{L}AJE\mathbb{1}_n\tilde{V}_{\text{PCC}}, \\ &\quad - \kappa\tilde{V}_{\text{PCC}}\mathbb{1}_n^\top B[V]A\mathcal{L}AQ_I - (\tilde{V}_{\text{PCC}})^2\mathbb{1}_n^\top BE\mathbb{1}_n \\ &= -\eta(V)^\top\Phi(V)\eta(V), \end{aligned} \tag{IV.15}$$

where Φ is defined in (IV.8) and

$$\eta(V) = \begin{bmatrix} A\mathcal{L}AQ_I \\ \tilde{V}_{\text{PCC}} \end{bmatrix} \quad \in \mathbb{R}^{n+1}. \tag{IV.16}$$

Since $\Phi(V^*) > 0$ by assumption, by continuity, there exists a (small) neighborhood around V^* such that $\Phi(V) > 0$. Thus, $\dot{\mathcal{F}} \leq 0$. Hence, V^* is a stable equilibrium point.

Furthermore, to establish asymptotic stability of V^*, we show that the condition

$$\Phi(V)\eta(V) = 0 \quad \Rightarrow \quad \lim_{t\to\infty} V(t) = V^* \tag{IV.17}$$

holds along the solutions of the system (IV.2). Since $\Phi(V) > 0$, (IV.17) implies that $\eta(V) = \mathbb{0}_{n+1}$. With $\eta(V)$ defined in (IV.16), $\eta(V) = \mathbb{0}_{n+1}$ is equivalent to $AQ_I(V) = \alpha\mathbb{1}_n, \alpha \in \mathbb{R}$ and $\tilde{V}_{\text{PCC}}(V) = 0$, which from Lemma 3.1 yields V^*. As a consequence, the invariant set where $\dot{\mathcal{F}} \equiv 0$ is the equilibrium of the system (IV.2). Therefore, V^* is locally asymptotically stable, completing the proof. ■

TABLE I: Line parameters of the simulated MG

Resistance R = 1.2 mΩ/km and reactance X = 9.5 mΩ/km				
Power line	$i = 1$	$i = 2$	$i = 3$	$i = 4$
Length (km)	2	5	7	3

V. Numerical Example

In this section, the performance of the control law (IV.2) is illustrated in simulation. At first, the MG employed is introduced and then the simulation scenario.

The parallel MG used in the case study has four inverters connected in parallel and is simulated using MATLAB®/Simulink® and PLECS [25]. The parameters used in the simulation are given in Table I. The base voltage and the base power values used in the pu (per-unit) calculation are $V_{\text{base}} = 20\text{kV}$ and $S_{\text{base}} = 4.75\text{MVA}$, respectively. The simulated MG contains four DGs, with nominal power rating $S_1^N = 1\text{pu}$, $S_2^N = 0.5\text{pu}$, $S_3^N = 0.33\text{pu}$ and $S_4^N = 0.25\text{pu}$. A small positive line resistance (see Table I) is considered in the simulated MG to evaluate robustness of (IV.2) towards typical modeling uncertainties. Furthermore, the DG phase angles are controlled by the standard frequency droop control [13]. Inspired by [14], the entries of the weighting matrix $A = \text{diag}(a_i)$ were chosen corresponding to the nominal power rating of each DG, i.e., $a_i = 1/S_i^N, i = 1, 2, 3, 4$. Finally, we have a constant current load $I_L = -0.9\text{pu}$ connected at the PCC. Since we are mainly interested in reactive power aspects, we assume that the active power demand of the constant current load is zero and hence $I_L = -0.9\text{pu}$ represents only the reactive power consumed at the PCC.

The Laplacian matrix used to implement (IV.2) is chosen as

$$\mathcal{L} = (100) \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Furthermore, the pinning gain matrix E was selected as $E = e\hat{E}$, where $e \in \mathbb{R}_{>0}$ is a controller parameter and $\hat{E} = \text{diag}(1, 0, 0, 0)$. The parameters e and κ are chosen as $e = 9.4$ and $\kappa = 0.005$.

The simulation results of the system (IV.2) using these parameters are shown in Figure 2. We observe that before 5sec, the PCC voltage is not at $V_{\text{PCC}}^d = 1\text{pu}$, see the enlarged plot before 5sec. Furthermore, as can be seen from the weighted reactive power injection (AQ_I) plot from 0 to 5sec, reactive power sharing is poor. The control law (IV.2) is activated at 5sec. After this, the PCC voltage converges quickly to 1pu, see the enlarged plot at 15sec. Furthermore, the weighted reactive power injections reach consensus, i.e., (II.9). This confirms that the DGs share the load I_L in a proportional fashion. An additional constant current load of -0.9pu is added at the PCC at 25sec. After this load jump, PCC voltage regulation (see the enlarged plot at 35sec) and reactive power sharing are swiftly re-established.

Furthermore, the condition $\Phi(V^*) > 0$ derived in Proposition 4.3 is verified for both operating points displayed in Fig. 2, i.e.,

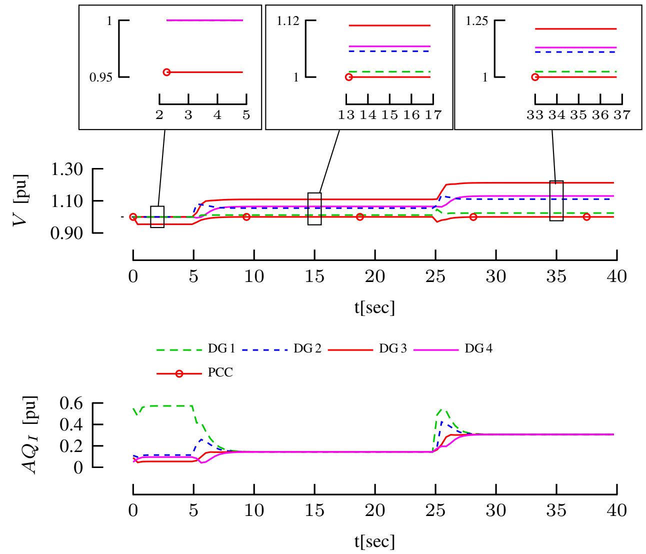

Fig. 2: Simulation result of (IV.2) with four inverters connected in parallel.

- $V_1^* = \text{col}(1.01, 1.05, 1.11, 1.06)$,
- $V_2^* = \text{col}(1.02, 1.11, 1.21, 1.13)$.

Hence, we conclude that these are both locally asymptotically stable. Also, as established in Lemma 3.1, it is straightforward to check that $V_i^* > V_{\text{PCC}}^d$. Therefore, the presented case study shows that the derived conditions are verified in a realistic example. In addition, the control performance with respect to load variations is also satisfactory.

VI. Conclusion

We have proposed a distributed voltage controller for parallel MGs which addresses the objectives of voltage regulation at the PCC and reactive power sharing. Furthermore, we have shown that for the case of an inductive constant current load connected at the PCC, the closed-loop system admits a unique positive voltage solution. A sufficient condition on the controller gains, which guarantees asymptotic stability of this desired solution was also presented. Finally, the numerical case study demonstrates that the proposed approach performs well in the event of load variations and in the presence of (small) line resistances.

In practice, the reactive power output of an inverter is measured through a low-pass filter [26]. Hence in future, we intend to incorporate these measurement dynamics also in the stability analysis. Furthermore, we plan to include different load models [18], [27] while designing voltage controllers for parallel MGs. Another interesting problem is the design of voltage controllers in networks with mixed R/X impedances.

References

[1] M. Chandorkar, D. Divan, and R. Adapa, "Control of parallel connected inverters in standalone AC supply systems," *IEEE Trans. Ind. Applicat.*, vol. 29, no. 1, pp. 136 –143, Jan/Feb 1993.

[2] Q.-C. Zhong, "Robust droop controller for accurate proportional load sharing among inverters operated in parallel," *IEEE Trans. Ind. Electron.*, vol. 60, no. 4, pp. 1281–1290, 2013.

[3] J. Guerrero, L. Garcia de Vicuna, J. Matas, M. Castilla, and J. Miret, "Output impedance design of parallel-connected UPS inverters with wireless load-sharing control," *IEEE Trans. Ind. Electron.*, vol. 52, no. 4, pp. 1126 – 1135, Aug. 2005.

[4] J. M. Guerrero, J. C. Vasquez, J. Matas, M. Castilla, and L. G. de Vicuna, "Control strategy for flexible microgrid based on parallel line-interactive UPS systems," *IEEE Trans. Ind. Electron.*, vol. 56, no. 3, pp. 726 – 736, March 2009.

[5] M. Marwali, J.-W. Jung, and A. Keyhani, "Control of distributed generation systems - Part II: Load sharing control," *IEEE Trans. Power Electron.*, vol. 19, no. 6, pp. 1551 – 1561, Nov. 2004.

[6] C. K. Sao and P. W. Lehn, "Autonomous load sharing of voltage source converters," *IEEE Trans. Power Delivery*, vol. 20, no. 2, pp. 1009–1016, 2005.

[7] J. Guerrero, J. Matas, L. G. de Vicuna, M. Castilla, and J. Miret, "Decentralized control for parallel operation of distributed generation inverters using resistive output impedance," *IEEE Trans. Ind. Electron.*, vol. 54, no. 2, pp. 994 –1004, April 2007.

[8] U. B. Jensen, F. Blaabjerg, and P. N. Enjeti, "Sharing of nonlinear load in parallel connected three-phase converters," *IEEE Trans. Ind. Applicat.*, vol. 37, no. 6, pp. 1817 – 1823, Nov/Dec 2001.

[9] A. Tuladhar, H. Jin, T. Unger, and K. Mauch, "Control of parallel inverters in distributed AC power systems with consideration of line impedance effect," *IEEE Trans. Ind. Applicat.*, vol. 36, no. 1, pp. 131 –138, Jan/Feb 2000.

[10] J. Guerrero, J. Matas, L. de Vicuna, M. Castilla, and J. Miret, "Wireless-control strategy for parallel operation of distributed-generation inverters," *IEEE Trans. Ind. Electron.*, vol. 53, no. 5, pp. 1461 –1470, Oct. 2006.

[11] W. Yao, M. Chen, J. Matas, J. Guerrero, and Z.-M. Qian, "Design and analysis of the droop control method for parallel inverters considering the impact of the complex impedance on the power sharing," *IEEE Trans. Ind. Electron.*, vol. 58, no. 2, pp. 576 –588, Feb. 2011.

[12] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "Voltage stabilization in microgrids via quadratic droop control," *IEEE Tran. on Automatic Control*, vol. 62, no. 3, Mar. 2017.

[13] J. Schiffer, R. Ortega, A. Astolfi, J. Raisch, and T. Sezi, "Conditions for stability of droop-controlled inverter-based microgrids," *Automatica*, vol. 50, no. 10, pp. 2457–2469, 2014.

[14] J. Schiffer, T. Seel, J. Raisch, and T. Sezi, "Voltage stability and reactive power sharing in inverter-based microgrids with consensus-based distributed voltage control," *IEEE Trans. Contr. Syst. Technol.*, vol. 24, no. 1, pp. 96 – 109, Jan. 2016.

[15] J. W. Simpson-Porco, Q. Shafiee, F. Dörfler, J. C. Vasquez, J. M. Guerrero, and F. Bullo, "Secondary frequency and voltage control of islanded microgrids via distributed averaging," *IEEE Trans. Ind. Electron.*, vol. 62, pp. 7025–7038, 2015.

[16] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[17] J. Schiffer, "Stability and power sharing in microgrids," Ph.D. dissertation, Technische Universität Berlin, 2015.

[18] P. Kundur, *Power system stability and control*. McGraw-Hill, 1994.

[19] J. Schiffer, D. Zonetti, R. Ortega, A. Stanković, J. Raisch, and T. Sezi, "Modeling of microgrids - from fundamental physics to phasors and voltage sources," *Automatica*, vol. 74, no. 12, pp. 135–150, 2016.

[20] C. Godsil and G. Royle, *Algebraic Graph Theory*. Springer, 2001.

[21] C. D. Persis and N. Monshizadeh, "Bregman storage functions for microgrid control," *IEEE Trans. Automat. Contr.*, 2017.

[22] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge university press, 2012.

[23] H. K. Khalil, *Nonlinear systems*. Prentice Hall, 2002, vol. 3.

[24] A. van der Schaft, *L2-Gain and Passivity Techniques in Nonlinear Control*. Springer, 2000.

[25] Plexim GmbH, "Plecs software, www.plexim.com," 2013.

[26] N. Pogaku, M. Prodanovic, and T. Green, "Modeling, analysis and testing of autonomous operation of an inverter-based microgrid," *IEEE Trans. Power Electron.*, vol. 22, no. 2, pp. 613 –625, march 2007.

[27] T. Van Cutsem and C. Vournas, *Voltage stability of electric power systems*. Springer, 1998, vol. 441.