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A Tool for Analysis of Existence of Equilibria and Voltage Stability in Power Systems with Constant Power Loads

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***Abstract*—It is well-known that constant power loads in power systems have a destabilizing effect. Their growing presence in modern installations significantly aggravates this issue, hence motivating the development of new methods to analyze their effect in AC and DC power systems. Formally, this problem can be cast as the analysis of solutions of a set of nonlinear algebraic equations of the form $f(x) = 0$, where $f : \mathbb{R}^n \to \mathbb{R}^n$, to which we associate the differential equation $\dot{x} = f(x)$. By invoking advanced concepts of dynamical systems theory and effectively exploiting its monotonicity, the following properties are established: (i) prove that, if there are equilibria, there is a *distinguished* one that is stable and attractive, and give conditions such that it is unique; (ii) give a simple on-line procedure to decide whether equilibria exist or not, and to compute the distinguished one; (iii) prove that the method is also applicable for the case when the parameters of the system are not exactly known. It is shown how the proposed tool can be applied to the analysis of long-term voltage stability in AC power systems, and to the study of existence of equilibria of multi-terminal high-voltage DC systems and DC microgrids.**

***Index Terms*—Power systems, constant power loads, existence of equilibria, voltage-stability .**

I. Introduction

A *sine qua non* condition for the correct operation of power systems is the existence of a steady-state behavior that, moreover, should be robust in the presence of perturbations [1]. The analysis of these equilibria is complicated by the presence of constant power loads (CPLs), which introduce "strong" nonlinearities and have a destabilizing effect that gives rise to significant oscillations or to network collapse. The growing presence of CPLs in modern AC and DC power systems—where they are used to model the behavior of some point-of-load converters—motivates the development of new methods to analyze these equilibria. It should be pointed out that the power systems community debates now new definitions of stability, which move away from the equilibrium-disturbance-equilibrium paradigm [2]. But, it is our belief that the analysis of equilibria will remain relevant in future AC and DC power systems; see, *e.g.*, [3] and [4].

In this paper we derive a methodological approach, which permits to determine existence and stability properties of equilibria in the following power systems problems.

- *P1* Analysis of *static* voltage stability of AC power systems with "light" active power load. The study of this important property, also called "long-term voltage stability" [1, Chapter 14], "loadability limit" [5, Chapter 7], or "voltage-regularity" [20], [21] is standard in the power systems community.[1] In the sequel, to avoid confusion with stability analysis in the sense of Lyapunov, we exclusively use the term voltage-regularity.
- *P2* Study of existence of steady-state behavior of two emerging power system concepts, namely multi-terminal high-voltage (MT-HV) DC networks [6], [7] and DC microgrids [8], [9].
- *P3* In addition, if stationary voltage solutions exist our method also allows to identify the solution with the highest voltage magnitudes, which is the desired operating condition in these applications.

In the three examples mentioned above, the key problem is the study of a nonlinear algebraic equation $f(x) = 0$, with $f : \mathbb{R}^n \to \mathbb{R}^n$, where only solutions x with positive components are of interest. The approach adopted in the paper is to associate to $f(x)$ the ordinary differential equation (ODE) $\dot{x} = f(x)$, which is well-defined on the positive orthant of $\mathbb{R}^n$, and to apply to it tools of dynamical systems [10]—in particular, monotone systems [11]—to study existence and stability of its equilibria, which are the solutions of the primal algebraic equation.

The main contributions of our work are the proofs of the following properties of the ODE.

- *C1.* If there are no equilibria (stable or unstable) then, in all solutions of the ODE, one or more components converge to zero in finite time.
- *C2.* If equilibria exist, there is a distinguished equilibrium, say $\bar{x}_{\mathtt{max}}$, among them that dominates component-wise all the other ones. This equilibrium $\bar{x}_{\mathtt{max}}$ attracts all trajectories starting in a certain well-defined domain. Moreover, we

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[1] The authors are indebted to Prof David Hill for pointing out the equivalence between these terms.

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give physically-interpretable conditions on the problem data that ensure $\bar{x}_{\mathtt{max}}$ is the only stable equilibrium.

C3. By solving a system of n convex algebraic inequalities in n positive unknowns we explicitly identify a set of initial states with the following characteristics: (i) all trajectories starting there monotonically decay in all components; (ii) either some component converges to zero in a finite time for all those trajectories or, for all of them, all components remain separated from zero on the infinite time horizon. Moreover, in the latter case, the trajectory is forward complete and converges to $\bar{x}_{\mathtt{max}}$. An additional outcome of this analysis is the generation of an estimate for the domain of attraction of asymptotically stable equilibia.

C4. Prove that the method is applicable even if the parameters of $f(x)$ are unknown, and only upper and lower bounds for them are available.

Comparison with existing literature: In [32], the authors propose conditions for the solvability of affinely parameterized quadratic equations that contain, as a particular case, the kind of nonlinear equations studied in the present manuscript. Nonetheless, a standing assumption in that paper is that *a solution exists*, and the focus is to derive conditions under which the solutions belong to a certain pre-specified set. Conversely, our paper does not take the existence for granted; instead we give conditions under which solutions exist (or not). Additionally, the identification of the dominant equilibrium $\bar{x}_{\max}$ and the analysis of its regularity properties—from the viewpoint of reactive power flow analysis—is carried out thanks to the stability identification of the ODE's equilibria; the latter central *analytical* aspect in our study cannot be addressed with the tools used in [32]. More recently, in [4], analytical conditions for the existence of solutions of the *full power flow* equations are given. However, by invoking the standard "decoupling" assumption, we address only the problem of reactive power flow. Finally, in [31], the authors propose a *numerical method* to solve the load flow equations, which is by now standard and implemented in many commercial software packages, such as *DigSilent PowerFactory*. Nevertheless, as we have already mentioned it, this problem is beyond the scope of our manuscript.

The remainder of the paper is organized as follows. Section II describes the ODE $\dot{x} = f(x)$ of interest and gives the main theoretical results pertaining to it, with some practical extensions given in Section III. In Section IV we illustrate these results with three canonical power systems examples. Section V presents some numerical simulation results. The paper is wrapped-up with concluding remarks in Section VI. To enhance readability, all proofs of the technical results are given in Appendices at the end of the paper.

Notation: Inequalities between vectors $x \in \mathbb{R}^n$ are meant component-wise, with $x_i \in \mathbb{R}$ its ith component and $|x| = \sqrt{x^\top x}$. The positive orthant of $\mathbb{R}^n$ is denoted as $\mathcal{K}_+^n := \{x \in \mathbb{R}^n : x > 0\}$, $\text{stack}(p_i) \in \mathbb{R}^{r_1+\cdots+r_N}$, denote s stacking $p_i \in \mathbb{R}^{r_i}, i \in \{1, \ldots, N\}$ on top of one another, $\text{diag}(A_1, \ldots, A_k)$, is the block-diagonal matrix composed of the listed square blocks A_i. All mappings are assumed smooth. Given a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ we denote its Jacobian by $\nabla f(x) := \frac{\partial f(x)}{\partial x}$. The operator $\langle\cdot\rangle$ denotes the clipping function $\langle a\rangle := \max\{a, 0\}$.

II. Analysis of the ODE of Interest

As indicated in the introduction, in this paper we are interested in the regularity of the voltage solutions of AC power systems (under the common decoupling assumption [1]), and in the study of the existence of steady-states for MT-HVDC networks as well as DC microgrids—the three of them with CPLs. In Section III, it is shown that these studies boil down to the analysis of solutions of the following algebraic equations in $\bar{x} \in \mathcal{K}_+^n$

$$a_{i1}\bar{x}_1 + a_{i2}\bar{x}_2 + \cdots + a_{in}\bar{x}_n + \frac{b_i}{\bar{x}_i} = w_i, \quad i \in \{1, \ldots, n\}, \tag{1}$$

where $a_{ij} \in \mathbb{R}$, $w_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$. These equations can be written in compact form as

$$A\bar{x} + \text{stack}\left(\frac{b_i}{\bar{x}_i}\right) - w = 0. \tag{2}$$

In the paper, we adopt the following.

Assumption 2.1: The matrix $A = A^\top$ is positive definite, its off-diagonal elements are non-positive and $b_i \neq 0$ for all i.

To study the solutions of (2) we consider the following ODE

$$\dot{x} = f(x) := -Ax - \text{stack}\left(\frac{b_i}{x_i}\right) + w. \tag{3}$$

We are interested in studying the existence, and stability, of the equilibria of (3). In particular, we will provide answers to the following questions.

Q1 When do equilibria exist? Is it possible to offer a simple test to establish their existence?

Q2 If there are equilibria, is there a distinguished element among them?

Q3 Is this equilibrium stable and/or attractive?

Q4 If it is attractive, can we estimate its domain of attraction?

Q5 Is it possible to propose a simple procedure to compute this special equilibrium using the system data (A, b, w)?

Q6 Are there other stable equilibria?

Instrumental to answer to the queries Q1—Q6 is the fact that the system (3) is monotone. That is, for any two solutions $x_a(\cdot), x_b(\cdot)$ of (3), defined on a common interval $[0, T]$, the inequality $x_a(0) \leq x_b(0)$ implies that $x_a(t) \leq x_b(t)\ \forall t \in [0, T]$. This can be verified by noticing that equation (3) satisfies the necessary and sufficient condition for monotonicity [11, Proposition 1.1 and Remark 1.1, Ch. III]

$$\frac{\partial f_i(x)}{\partial x_j} \geq 0,\ \forall i \neq j,\ \forall x \in \mathcal{K}_+^n. \tag{4}$$

In the sequel, we denote by $x(t, x_0)$ the solution of (3) with initial condition $x(0) = x_0 > 0$, and use the following.

Definition 2.1: An equilibrium $\bar{x} > 0$ of (3) is said to be attractive from above if for any $x_0 \geq \bar{x}$, the solution $x(t, x_0)$ is defined on $[0, \infty)$ and converges to $\bar{x}$ as $t \to \infty$. The equilibrium is said to be hyperbolic if the Jacobian matrix $\nabla f(\bar{x})$ has no eigenvalue with zero real part [10].

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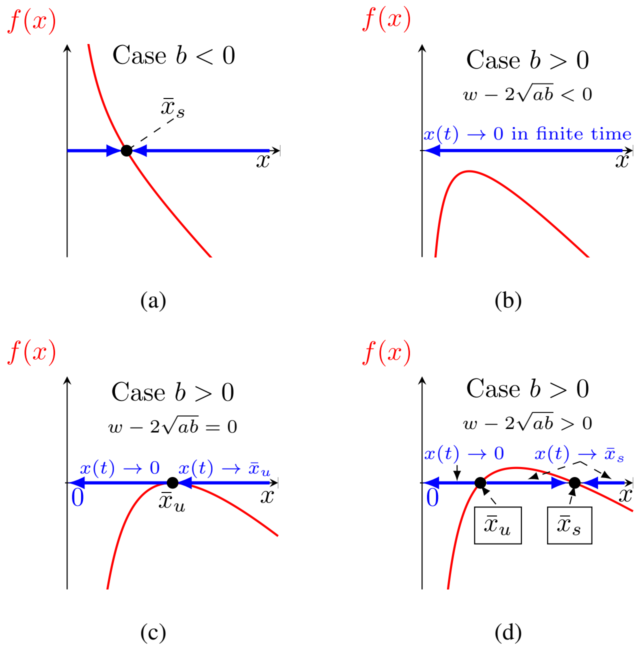

Fig. 1: Feasible behaviors of the one-dimensional system (5): (a) A unique globally attractive equilibrium $\bar{x}_s$; (b) No equilibria, all solutions converge to zero in a finite time t_f; (c) Unique unstable equilibrium $\bar{x}_u$, which is attractive from above, whereas any solution starting on the left diverges from $\bar{x}_u$ and converges to 0 in a finite time; (d) Two equilibria, the smallest of which $\bar{x}_u$ is unstable, whereas the larger one $\bar{x}_s$ is stable and attractive from above.

Remark 2.1: Along the lines of the standard Kron reduction, the findings of the paper can be extended to the case where some of the coefficients b_i are equal to zero.[2]

The proof of this remark is given in Appendix A.

A. The simplest example

To gain an understanding of some key traits of possible results, it is instructive to start with the simplest case $n = 1$. Then, $x \in \mathbb{R}$ and (3) is the scalar equation

$$\dot{x} = -ax - \frac{b}{x} + w, \tag{5}$$

where $a > 0, b \neq 0$. Feasible behaviors of the system are exhaustively described in Figure 1.

The following can be inferred from this figure.

- *B1)* The system either has no equilibria, or it has finitely many equilibria, or it has a single equilibrium.
- *B2)* If the system has equilibria, the rightmost of them is attractive from above.
- *B3)* Non-hyperbolic equilibria may be attractive from above but unstable; moreover, there may be no other equilibria.
- *B4)* Hyperbolic and attractive from above equilibria are stable.
- *B5)* If $b > 0$, globally attractive equilibria do not exist.

We will show below that several of the traits mentioned above are inherited by the n-th order ODE (3).

[2]We thank an anonymous Reviewer for indicating this.

B. Main results on the system (3)

This section offers a qualitative analysis of the system (3); the proofs of the respective results are placed in Appendix D.

Proposition 2.1: Consider the system (3) verifying Assumption 2.1. One and only one of the following two mutually exclusive statements holds.

- *S1)* There are no equilibria $\bar{x}$, either stable or unstable, and any solution $x(\cdot)$ is defined only on a finite time interval $[0, t_f) \subset [0, \infty)$, since there exists at least one coordinate x_i such that $x_i(t) \to 0, \dot{x}_i(t) \to -\infty$ as $t \to t_f$. Such a coordinate is necessarily associated with $b_i > 0$.[3]
- *S2)* Equilibria $\bar{x}^k$ do exist. One of them $\bar{x}_{\mathtt{max}} > 0$ verifies $\bar{x}_{\mathtt{max}} \geq \bar{x}^k$, $\forall k$, and is attractive from above.

□□□

We remark that a situation similar to B3) is undoubtedly feasible for multidimensional systems as well —*e.g.*, considering a diagonal matrix A. Nevertheless, now we'll give evidence suggesting that this situation "almost never" occurs. This forms a ground to treat the opposite situation as "typical" or "prevalent" and to pay special attention to it. We also specifically identify a case where B3) does not occur, as well as the associated implications regarding equilibria cardinality and stability.

Proposition 2.2: Consider again the system (3) that satisfies Assumption 2.1. If, in addition,

$$\left\{ x \in \mathcal{K}_+^n \;\middle|\; \begin{array}{c} \det\left[A - \mathrm{diag}\left(\frac{b_i}{x_i^2}\right)\right] = 0, \\ w = Ax + \mathrm{stack}\left(\frac{b_i}{x_i}\right) \end{array} \right\} = \emptyset, \tag{6}$$

then the following statements are true:

- *Y1)* If equilibria of (3) do exist, then there are finitely many of them and $\bar{x}_{\mathtt{max}}$ is locally asymptotically stable.
- *Y2)* If all the coefficients b_i are of the same sign, then there are no other stable equilibria apart from $\bar{x}_{\mathtt{max}}$.

Meanwhile, the identity (6) holds for almost all parameters of the system (3), as is rigorously specified by the following.

- *Y3)* For any given A and $b_i \neq 0$, the set of all $w \in \mathbb{R}^n$ for which (6) *does not* hold has zero Lebesgue measure and is nowhere dense.

□□□

The claim Y3) allows to assert that the claims Y1) and Y2) are true almost surely. The condition (6) is necessarily met if $b_i < 0\ \forall i$ since then the matrix $A - \mathrm{diag}\left(\frac{b_i}{x_i^2}\right)$ is positive definite due to Assumption 2.1 and so the "upper" equation in (6) has no roots. In the case of a diagonal matrix A, the condition (6) is necessary for $\bar{x}_{\mathtt{max}}$ to be locally asymptotically stable, as is shown by arguments in subsection II-A. Full elaboration of the issue of local stability of $\bar{x}_{\mathtt{max}}$ in the "zero measure case" where (6) does not hold is beyond the scope of this paper and is a topic of our ongoing research. In the context of power system analysis, points at which (6) is satisfied are called *loadability limits* [5, Chapter 7] or *stability limits* [1, Chapter 14]. From a physical perspective, these are

[3]This implies that the case S1) does not occur if $b_j < 0, \forall\, j$.

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points where the load attains a maximum feasible value after which no further equilibrium solutions exist [5], Chapter 7.

Now we are going to offer a constructive test to identify which of the cases S1) or S2) holds, as well as a method to find $\bar{x}_{\mathtt{max}}$ in the case S2). To this end, we introduce the following.

Definition 2.2: A solution $x(\cdot)$ of the ODE (3) is said to be *distinguished* if its initial condition lives in the set

$$\mathcal{E} := \left\{ x \in \mathcal{K}_+^n \mid Ax > \text{stack}\left(\langle w_i \rangle + \frac{\langle -b_i \rangle}{x_i} \right) \right\}. \quad (7)$$

If all coefficients $b_i > 0$, the set (7) reduces to the (convex open polyhedral) cone $\left\{ x \in \mathcal{K}_+^n \mid Ax > \text{stack}\left(\langle w_i \rangle \right) \right\}$.

Proposition 2.3: Consider the system (3) verifying Assumption 2.1. Then the following statements are true:

I) The set $\mathcal{E}$ is non-empty, consequently there are distinguished solutions.

II) All such solutions strictly decay, in the sense that $\dot{x}(t) < 0$, for all t in the domain of definition of $x(\cdot)$.

III) One and only one of the following two mutually exclusive statements holds simultaneously for all distinguished solutions $x(\cdot)$:

(i) For a finite time $t_f \in (0, \infty)$, some coordinate $x_i(\cdot)$ approaches zero, that is,

$$\lim_{t \to t_f} x_i(t) = 0, \quad (8)$$

and the solution $x(\cdot)$ is defined only on a finite time interval $[0, t_f)$.

(ii) There is no coordinate approaching zero, the solution is defined on $[0, \infty)$, and the following limit exists and verifies

$$\lim_{t \to \infty} x(t) > 0. \quad (9)$$

This limit is the same for all distinguished solutions.

IV) If the case *(i)* holds for a distinguished solution, the situation *S1)* from Proposition 2.1 occurs.

V) If the case *(ii)* holds for a distinguished solution, the situation *S2)* from Proposition 2.1 occurs, and the dominant equilibrium $\bar{x}_{\mathtt{max}}$ is equal to the limit (9). □□□

The proof of this proposition is given in Appendix D.

C. *Some additional properties of the system* (3)

P1) In *III.i)*, there may be several components x_i with the described property, but not all of them necessarily possess it.

P2) The claim *S1)* in Proposition 2.1 and *IV* in Proposition 2.3 yield that (8) is necessarily associated with $b_i > 0$ and $\dot{x}_i(t) \to -\infty$ as $t \to t_f$.

P3) Regarding the claim *S2)* in Proposition 2.1 the basin of attraction of the equilibrium $\bar{x}_{\mathtt{max}}$ contains all states $x \geq \bar{x}_{\mathtt{max}}$. Under the condition (6) in Proposition 2.2, this basin is open.

P4) The linear programming problem of finding elements in the set $\left\{ x \in \mathcal{K}_+^n \mid Ax > 0 \right\}$ has been widely studied in the literature [12], [13], [14]. There is a whole variety of computationally efficient methods to solve this problem, including the Fourier-Motzkin elimination, the simplex method, interior-point/barrier-like approaches, and many others; for a recent survey, we refer the reader to [15].

III. A Numerical Procedure and a Robustness Analysis

With the aim of extending the realm of application of the previous results, we address in this section the issues of numerical computation and robustness of the claims. We also give an explicit answer to the questions raised at the beginning of Section II.

A. *A numerical procedure to verify Proposition 2.3*

Proposition 2.3 suggests a computational procedure to verify whether the system has equilibria and, if they do exist, to find the dominant one $\bar{x}_{\mathtt{max}}$ among them. Specifically, it suffices to find an element of the set $\mathcal{E}$ defined in (7), to launch the solution of the differential equation (3) from this vector, and to check whether—as the solution decays—there is a coordinate approaching zero or, conversely, all of them remain separated from zero. In the last case, the solution has a limit, which is precisely the dominant equilibrium of the system.

The statement I) of Proposition 2.3 ensures that the first step of this algorithm, *i.e.*, generating an element of the set $\mathcal{E}$ defined in (7), is feasible. Technically, this step consists in solving the following system of feasible convex inequalities:

$$\langle w_i \rangle + \frac{\langle -b_i \rangle}{x_i} - \sum_{j=1}^{n} a_{ij} x_j < 0, \quad x_i > 0, \qquad \forall i.$$

This problem falls within the area of convex programming and so there is an armamentarium of effective tools to solve it. Nevertheless, this problem can be further simplified via transition from nonlinear convex inequalities to linear ones, modulo closed-form solution of finitely many scalar quadratic equations. The basis for this is given by the following lemma, whose proof is given in Appendix B.

Lemma 3.1: Pick any vector $z > 0$ such that $Az > 0$.[4] Then, the scaled vector $x := \mu z$ verifies $x \in \mathcal{E}$, provided that

$$\mu > \frac{\langle w_i \rangle + \sqrt{\langle w_i \rangle^2 + 4(Az)_i \frac{\langle -b_i \rangle}{z_i}}}{2(Az)_i}, \ \forall i. \quad (10)$$

For any i with $b_i > 0$, relation (10) simplifies into

$$\mu > \frac{\langle w_i \rangle}{(Az)_i}.$$

B. *Robustness* vis-à-vis *uncertain parameters*

The proposition below extends the results of Proposition 2.1 to the case where the parameters $\mathfrak{C} := (A, \{b_i\}_i, w)$ of the system (2) are not known, but only their component-wise bounds are available

$$A^+ \leq A \leq A^-, \quad b_i^+ \leq b_i \leq b_i^-, \quad w^- \leq w \leq w^+. \quad (11)$$

[4] In Appendix B it is shown that, under Assumption 2.1, this system of linear inequalities is feasible; see Lemma B.1 .

To streamline the presentation of the proposition we define the bounding sets $\mathfrak{C}^{\pm} := (A^{\pm}, \{b_i^{\pm}\}_i, w^{\pm})$.

Proposition 3.1: Suppose that $\mathfrak{C}$ verifies Assumption 2.1, and (11) holds. Then, the following statements are true.

i) If the case *S1)* from Proposition 2.1 holds for $\mathfrak{C}^+$, this case also holds for both $\mathfrak{C}$ and $\mathfrak{C}^-$;

ii) If the case *S2)* from Proposition 2.1 hold for $\mathfrak{C}^-$, this case also holds for both $\mathfrak{C}$ and $\mathfrak{C}^+$;

iii) In the situation *ii)*, the dominant equilibria $\bar{x}_{\max}^-$, $\bar{x}_{\max}$, $\bar{x}_{\max}^+$ related to $\mathfrak{C}^-, \mathfrak{C}, \mathfrak{C}^+$, respectively, are such that

$$\bar{x}_{\max}^- \leq \bar{x}_{\max} \leq \bar{x}_{\max}^+. \tag{12}$$

Moreover, $x_{\max}^+ \to x_{\max}^-$ as $A^+ \to A^-, b_i^+ \to b_i^-, w^+ \to w^-$. □□□

The proof of this proposition is given in Appendix D.

Proposition 3.1 can be used to estimate the distance from the initial state $x \in \mathcal{E}$ of the employed distinguished solution to the dominant equilibrium $\bar{x}_{\max}$. This can be done picking $\mathfrak{C}^-$ and noting that, due to (12) and II) in Proposition 2.3, the distance of interest does not exceed $\|x - \bar{x}_{\max}^-\|$. Certainly, $\mathfrak{C}^-$'s with easily computable $\bar{x}_{\max}^-$ are of especial interest. An example is obtained via "zeroing" all off-diagonal elements of A in $\mathfrak{C}$ if for any i either $b_i < 0$ or $b_i > 0$ and $w_i \geq 2\sqrt{a_{ii} b_i}$. Then

$$x_{\max}^- = \text{stack}\Big(\frac{w_i + \sqrt{w_i^2 - 4a_{ii}b_i}}{2a_{ii}}\Big).$$

Thanks to S2) in Proposition 2.1 and (12), $\bar{x}_{\max}^-$ can be used, instead of a vector from the set (7), as the initial state when seeking $\bar{x}_{\max}$ via integration of the ODE (3). An example of this situation, happens when $b_i > 0 \; \forall i$. Then, in (11), we can take $A^+ := A, w^+ := w$, and $b_i^+ > 0$ arbitrarily close to 0, and moreover, finally let $b_i^+ \to 0+$. Then, by using Lemma 3.1, it can be shown that integration of the ODE can be started with $A^{-1}w$ provided that $A^{-1}w > 0$.[5]

C. *Answers to the queries* Q1–Q6 *in Section II*

Now we put the previous discussion of this section into a nutshell by giving a synopsis of our answers to *Q1–Q6*.

Answer to Q1

- Equilibria exist if and only if neither distinguished solution of the ODE (3) approaches the boundary of the positive cone $\mathcal{K}_+^n$ for a finite time;
- to check this existence criterion, it suffices to examine the behavior of an arbitrary distinguished solution;
- to accomplish the latter, a solution $x \in \mathbb{R}^n$ for a certain convex or a less conservative linear programming problem should be found and then the ODE (3) should be integrated from this vector x.

Answer to Q2

- There exists a distinguished equilibrium, $\bar{x}_{\max}$, that satisfies $\bar{x}_{\max} \geq \bar{x}_{\text{eq}}$, for any other equilibrium, $\bar{x}_{\text{eq}}$.

Answer to Q3

- This equilibrium is attractive from above and "almost surely" is stable and locally asymptotically stable.

[5]A rigorous proof of this fact is omitted for brevity.

Answer to Q4

- This domain covers the set (7); the latter set contains any vector built as is discussed in Lemma 3.1.

Answer to Q5

- It suffices to invoke the solution of the ODE (3) from the answer to *Q1* and to compute it until it converges.

Answer to Q6

- If in (2), all b_i's are of the same sign and the condition (6) holds, there are no other stable equilibria.

IV. Application to Some Canonical Power Systems

In this section we apply the results of Section II to three different problems of power systems containing constant power loads. These comprise the standard analysis of voltage-regularity of AC power systems with "light" active power loading as well as the study of existence of equilibria of MT-HVDC networks and DC microgrids.

A. *Voltage-regularity in AC power systems*

The standard static analysis of voltage stability in AC power systems assumes the dynamics is in steady-state, and concentrates its attention on the algebraic equations relating the active and reactive power, with the voltages and the phase angles—the well-known power flow equations. In [16] it was first suggested to investigate the sign of the real parts of the eigenvalues of the power flow Jacobian matrix as an indicator of voltage stability. This sensitivity analysis of the voltage magnitudes with respect to changes in the active and reactive power flows is the prevailing approach to analyze voltage stability in AC networks as explained in power systems textbooks, *i.e.*, [1, Chapter 14] and [5, Chapter 7]. In this subsection we show how the analysis framework developed in Section II can be applied to carry out this analysis, for the particular case of power systems with "light" active power load.

Consider a high-voltage AC power network with $n \geq 1$ nodes. Denote by $V_i > 0$, δ_i and Q_i the voltage magnitude, phase angle and the reactive power load demand at node i, respectively. We restrict our analysis to scenarios in which the following standard "light" active power load (also called "decoupling") assumption is satisfied [1, Chapter 14.3.3], [17, Assumption 1].

Assumption 4.1: $\delta_i - \delta_j \approx 0$ for all $i, j \in \{1, \dots, n\}$.

With Assumption 4.1, the reactive power flow at node i is given by [1], [18], [17] (see [19, Section 5] for a detailed derivation of the power flow equations)

$$Q_{\text{ZIP},i} = V_i \sum_{j=1}^{n} |B_{ij}|(V_i - V_j),$$

where $B_{ij} < 0$ if nodes i and j are connected via a power line and $B_{ij} = 0$ otherwise. The reactive power demand $Q_{\text{ZIP},i}$ at the i-th node is described by a, so-called, ZIP model, namely

$$Q_{\text{ZIP},i} := \left(\mathcal{Y}_i V_i^2 + k_i V_i + Q_i\right).$$

The term ZIP load refers to a parallel connection of a constant susceptance $\mathcal{Y}_i \in \mathbb{R}$, a constant reactive current $k_i \in \mathbb{R}$, and a constant power $Q_i \in \mathbb{R}$ load. Then, we obtain the (algebraic) reactive power balance equation

$$\left(\mathcal{Y}_i V_i^2 + k_i V_i + Q_i\right) = V_i \sum_{j=1}^{n} |B_{ij}|(V_i - V_j),\ i \in \{1, \ldots, n\}, \tag{13}$$

which by defining $x := \text{stack}\,(V_i) \in \mathcal{K}_+^n$, $A \in \mathbb{R}^{n\times n}$ with

$$\begin{aligned} A_{ii} &= \sum_{j=1}^{n} |B_{ij}| - \mathcal{Y}_i,\ A_{ij} = -|B_{ij}|, \\ w &= \text{stack}(k_i),\ b_i = -Q_i, \end{aligned} \tag{14}$$

can be rewritten as (2). If we make the reasonable assumption that $\mathcal{Y}_i < 0$ for at least one node, the matrix A satisfies Assumption 2.1.

We introduce the following notion of voltage-regularity—which we recall is also known as static (or long-term) voltage stability—for the system (2) with the parameters (14), which relates the analysis of Section II to standard power system practice, see [1, Chapter 14], [5, Chapter 7] and the more recent work [18].

Definition 4.1 (cf. [20], [21]): A positive root $\bar{x}$ of the system (2) is voltage-regular if the Jacobian $\nabla f(x)\big|_{x=\bar{x}}$, with $f(x)$ given in (3) with the parameters (14), is Hurwitz.

The following remarks concerning the application of the results of Section II to this particular problem are in order.

R1) The coefficients $-b_i$ are the constant reactive powers extracted or injected into the network, being positive (capacitive) in the former case, and negative (inductive) in the latter. As indicated in Section II, sharper results—*i.e.*, that $\bar{x}_{\texttt{max}}$ is the only stable equilibrium, and a simpler structure of the set $\mathcal{E}$ of initial conditions for the distinguished solutions—may be available if the signs of the coefficients b_i are the same and condition (6) is verified. Hence, the proposed conditions have a direct interpretation in terms of reactive power demand.
R2) The solution $\bar{x}_{\texttt{max}}$ for the system (13) represents the physically admissible steady state for the network with the highest values of voltage magnitudes at each node, which is the usually desired high-voltage operating point.
R3) Lemma C.2 in the Appendix implies that the Jacobian of the dynamics (3) evaluated at any stable equilibrium point is Hurwitz. Hence, if case V) of Proposition 2.3 applies then the dominant equilibrium is voltage-regular in the sense of Definition 4.1. Consequently, Proposition 2.3 provides a constructive procedure to evaluate the existence of a unique dominant and voltage-regular solution in power systems with constant power loads.

B. Multi-terminal HVDC transmission networks with constant power devices

An MT-HVDC network with n power-controlled nodes ($\mathcal{P}$-nodes) and s voltage-controlled nodes ($\mathcal{V}$-nodes), interconnected by m RL transmission lines, can be modeled by [22]

$$\begin{aligned} \tau \dot{I}_t &= -I_t - h(V), \\ L\dot{I} &= -RI + \mathcal{B}_{\mathcal{P}}^\top V + \mathcal{B}_{\mathcal{V}}^\top V_{\mathcal{V}}, \\ C\dot{V} &= I_t - \mathcal{B}_{\mathcal{P}} I - GV, \end{aligned} \tag{15}$$

TABLE I: Nomenclature for the model (15).

	State variables
I_t	$\mathcal{P}$-nodes injected currents
V	$\mathcal{P}$-nodes voltages
I	Line currents
	Parameters
L	Line inductances
C	$\mathcal{P}$-nodes shunt capacitances
R	Line resistances
G	$\mathcal{P}$-nodes shunt conductances
τ	Converter time constants
$V_{\mathcal{V}}$	$\mathcal{V}$-nodes voltages

where $I_t \in \mathbb{R}^n$, $V \in \mathcal{K}_+^n$, $I \in \mathbb{R}^m$ and $V_{\mathcal{V}} \in \mathbb{R}^s$. The matrices R, L, G, C, and τ are diagonal, positive definite of appropriate sizes. The physical meaning of each state variable and of every matrix of parameters is given in Table I. Furthermore, $\mathcal{B} = \text{stack}\,(\mathcal{B}_{\mathcal{V}}, \mathcal{B}_{\mathcal{P}}) \in \mathbb{R}^{(s+n)\times m}$ denotes the, appropriately split, node-edge incidence matrix of the network. The open-loop current injection at the power terminals is described by

$$h(V) = \text{stack}\left(\frac{P_i}{V_i}\right),$$

where $P_i \in \mathbb{R}$ denotes the power setpoint.[6] See also [23] for a systematic model procedure of HVDC systems using the port-Hamiltonian framework.

It can be shown that (15) admits an equilibrium if and only if the algebraic equations

$$0_n = -h(\bar{V}) - \left(\mathcal{B}_{\mathcal{P}} R^{-1} \mathcal{B}_{\mathcal{P}}^\top + G\right)\bar{V} - \mathcal{B}_{\mathcal{P}} R^{-1} \mathcal{B}_{\mathcal{V}}^\top V_{\mathcal{V}}, \tag{16}$$

have real solutions for $\bar{V} \in \mathcal{K}_+^n$. Notice that (16) is equivalent to the right hand side of (3) if we define $x := \bar{V}$ and

$$A := \mathcal{B}_{\mathcal{P}} R^{-1} \mathcal{B}_{\mathcal{P}}^\top + G,\ b_i := P_i,\ w := -\mathcal{B}_{\mathcal{P}} R^{-1} \mathcal{B}_{\mathcal{V}}^\top V_{\mathcal{V}}.$$

Given that $\mathcal{B}_{\mathcal{P}}$ is an incidence matrix, R and G are diagonal positive definite matrices, then, the term $\mathcal{B}_{\mathcal{P}} R^{-1} \mathcal{B}_{\mathcal{P}}^\top$ is a Laplacian matrix and thus it is positive semidefinite. Consequently, $A = A^\top$ is positive definite and Assumption 2.1 is satisfied and the results of Section II can be used to analyze the existence of equilibria of the dynamical system (15). This, through the computation of the solutions of $\dot{x} = f(x)$, taking f as the right hand side of (16).

As in Remark R1) of the previous example, the coefficients $-b_i$ are the powers extracted or injected into the network, being negative in the former case and positive in the latter. The observation of Remark R2) is also applicable in this example.

[6]The first equation in (15) represents the simplified converter dynamics, see [22, Section II, equation (18)] and [22, Figure 4]. The converter usually has a PI current control, see the equations (27) and (28) of [22]. For simplicity, we chose to study equilibria of the network without the PI. Nonetheless, our methodology applies also to the closed-loop scenario.

TABLE II: Nomenclature for the model (17).

	State variables
I_t	Generated currents
V	Load and bus voltages
I	Line currents
	Parameters
L_t	Filter inductances
L	Line inductances
C	Shunt capacitances
R_t	Filter resistances
R	Line resistances
	External variables
u	Control input (converter voltage)
I_{ZIP}	$\mathcal{Y}_i$: Constant impedance k_i: Constant current P_i: Constant power

C. DC microgrids with constant power loads

A standard Kron-reduced model of a DC microgrid, with $n \geq 1$ converter-based distributed generation units, interconnected by $m \geq 1$ RL transmission lines, can be written as [24]

$$\begin{aligned} L_t \dot{I}_t &= -R_t I_t - V + u, \\ C_t \dot{V} &= I_t + \mathcal{B} I - I_{\text{ZIP}}(V), \\ L \dot{I} &= -\mathcal{B}^\top V - RI, \end{aligned} \tag{17}$$

where $I_t \in \mathbb{R}^n$, $V \in \mathcal{K}_+^n$, $u \in \mathcal{K}_+^n$ and $I \in \mathbb{R}^m$ as well as R_t, R, L_t, L and C_t are diagonal, positive definite matrices of appropriate size; the physical meaning of each term appears in Table II. We denote by $\mathcal{B} \in \mathbb{R}^{n\times m}$, with $\mathcal{B}_{ij} \in \{-1, 0, 1\}$, the node-edge incidence matrix of the network. The load demand is described by a ZIP model, *i.e.,*

$$I_{\text{ZIP}}(V) = \mathcal{Y}V + k + \text{stack}\left(\frac{P_i}{V_i}\right),$$

where $\mathcal{Y} \in \mathbb{R}^{n\times n}$ is a diagonal positive semi-definite matrix, $k \in \mathbb{R}^n$ is a constant vector, and $P_i \in \mathbb{R}$.

Some simple calculations show that, for a given $u = \bar{u}$ constant, the dynamical system (17) admits a real steady state if and only if, the algebraic equations

$$0_n = R_t^{-1}\left(\bar{u} - \bar{V}\right) - \mathcal{B}R^{-1}\mathcal{B}^\top \bar{V} - I_{\text{ZIP}}(\bar{V}), \tag{18}$$

have real solutions for $\bar{V} \in \mathcal{K}_+^n$. Defining $x := \bar{V}$ and

$$A := R_t^{-1} + \mathcal{Y} + \mathcal{B}R^{-1}\mathcal{B}^\top,\ b_i := P_i,\ w := R_t^{-1}\bar{u} - k.$$

the system (18) can be written in the form (2). Similarly as for the MT-HVDC model, it can be shown that A is a positive definite matrix and, hence, satisfies the conditions in Assumption 2.1. Therefore, the results of Section II can be applied to study the solutions of the steady-state equation (18).

Once again, Remarks R1) and R2) of Subsection IV-A are also applicable here.

V. Numerical simulations

In this section we present some numerical simulations that illustrate the results reported in Section II.

A. An RLC circuit with constant power loads

The electrical network shown in Fig. 2 has been used in [25] as a benchmark example to study the existence of its equilibria. In steady state, this network is described by the system of quadratic equations

$$\begin{aligned} z &= -Yv + u, \\ v_i z_i &= P_i > 0,\ i = 1, 2, \end{aligned} \tag{19}$$

where z_i is the inductor's L_i current, v_i is the capacitor's C_i voltage, P_i is the power of the $i-$th CPL, and

$$Y = \begin{bmatrix} \frac{1}{r_2} + \frac{1}{r_1} & -\frac{1}{r_2} \\ -\frac{1}{r_2} & \frac{1}{r_2} \end{bmatrix},\ u = \begin{bmatrix} \frac{E}{r_1} \\ 0 \end{bmatrix}.$$

Define

$$x := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},\ A := Y,\ b_i = P_i,\ w := u,$$

then, the algebraic system (19) can be equivalently written in the form of (2), and hence its solvability can be studied via computing distinguished solutions, $x(t, x_0)$, of the ODE (3).

To analyze the system using Proposition 2.3, first we need to identify the set $\mathcal{E}$ (see equation (7)), which for this example is given by

$$\mathcal{E} = \left\{x \in \mathcal{K}_+^n \mid \left(\frac{1}{r_2} + \frac{1}{r_1}\right)x_1 - \frac{1}{r_2}x_2 > \frac{E}{r_1},\ -\frac{1}{r_2}x_1 + \frac{1}{r_2}x_2 > 0\right\},$$

or in a simpler form by

$$\mathcal{E} = \left\{x \in \mathcal{K}_+^n \mid E < x_1 < x_2 < \frac{(r_1 + r_2)}{r_1}x_1 - \frac{r_2 E}{r_1}\right\}.$$

This set is illustrated in Fig. 3a for the parameters' values of Table III; a distinguished solution of the ODE $\dot{x} = f(x)$ is also plotted.

We test our method in three steps: first, we take the initial condition $x_0 = (25.01, 25.77)$, which belongs to the set $\mathcal{E}$; then, we fix two different values for CPLs' powers, which we recall that are codified by the vector b; and finally, for each of these values of b, we compute the associated distinguished solution and observe its behavior.

Let us first fix $b = (500, 450)$. The plot of the associated distinguished solution is shown in Fig. 3c; notice that none of its components is approaching to zero, hence, we have the case *III.(ii)* of Proposition 2.3: the network admits equilibria. Furthermore, $x(t, x_0)$ asymptotically converges to $\bar{x}_{\texttt{max}} = (22.24, 20.95)$. The latter equilibrium is the only ODE's stable equilibrium, a fact which is established from Proposition 2.2 by observing that $b_i > 0$ for all i.

The described procedure is repeated now fixing $b = (3000, 1000)$. The plot of the associated distinguished solution is shown in Fig. 3d; its second component, denoted by $x_2(t)$, converges to zero in finite time, hence, we fall in the scenario *III.(i)* of Proposition 2.3: the network has no equilibria.

A graphical comparison that clearly illustrates the radically different behavior between the former and the latter distinguished solutions is shown in Fig. 3e.

Finally, we underscore the consistency of our method with respect to the *analytical* necessary and sufficient condition for existence of equilibria presented in [25, Proposition 1 and 3]: the shadowed region shown in Fig. 3b, which represents the values of b for which equilibria exist, can be obtained using this condition.

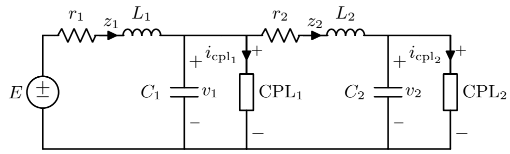

Fig. 2: DC Linear RLC circuit with two CPLS

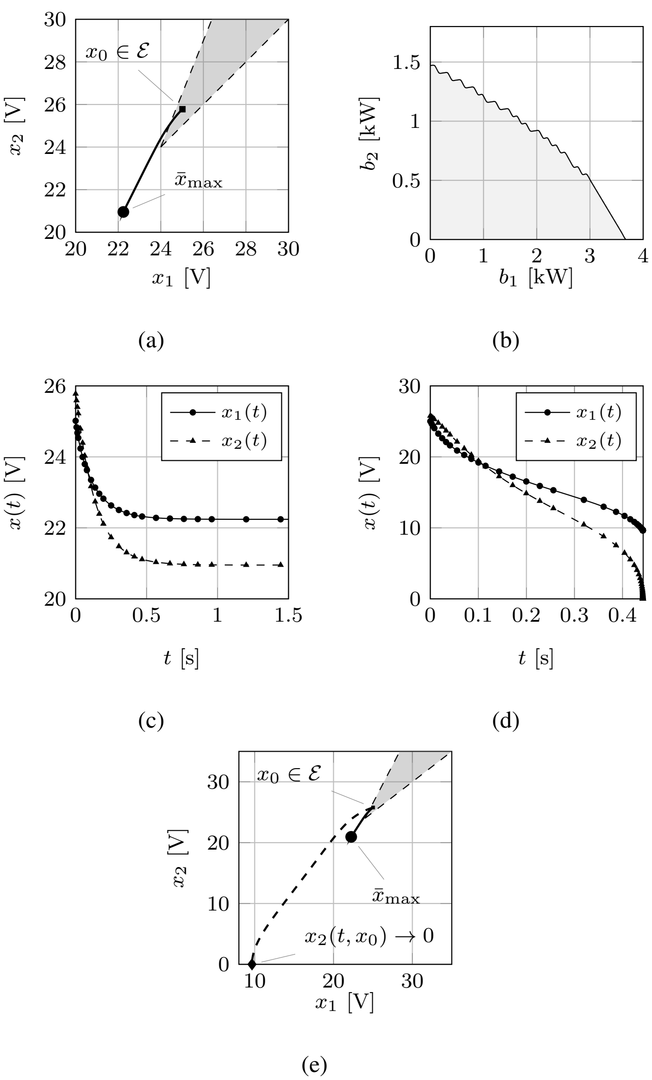

Fig. 3: Simulation results for the RLC circuit of Fig. 2: (a) plot of a a portion of the set $\mathcal{E}$ and a distinguished solution converging to $\bar{x}_{\mathtt{max}}$. (b) Set of positive values (shadowed region) for (b_1, b_2) for which the network admits an equilibrium. (c) Distinguished solution $x(t, x_0)$, with $b = (500, 450)$, converging to the equilibrium point $\bar{x}_{\mathtt{max}}$. (d) Distinguished solution $x(t, x_0)$, taking $b = (3000, 1000)$, with one of its components converging to zero in finite time: the system has no equilibrium points. (e) Phase-space plot of the distinguished solution $x(t, x_0)$ for two different values of b: one feasible and another one infeasible. Convergence to $\bar{x}_{\mathtt{max}}$ is observed in the former (solid curve), and convergence of the second component to zero is visualized in the latter (dashed curve).

TABLE III: Simulation Parameters of the multi-port network of Fig. 2.

E (V)	r_1 (Ω)	L_1(μH)	C_1(mF)
24	0.04	78	2
	r_2 (Ω)	L_2(μH)	C_2(mF)
	0.06	98	1

TABLE IV: Numerical parameters associated with the edges for the network in Fig. 4.

Transmission line	c_1	c_2	c_3	c_4	c_5
r_i (Ω)	0.9576	1.4365	1.9153	1.9153	0.9576

B. An HVDC transmission system

In this subsection we numerically evaluate the existence (and approximation) of equilibrium points for the particular HVDC system presented as an example in [22, Fig. 5]. The network, whose associated graph is shown in Fig. 4, consists in four nodes $\mathcal{N} = \{\mathcal{V}_1, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$, where $\mathcal{V}_1$ is a voltage controlled node, with voltage $V_{\mathcal{V}}^{(1)} = E$, and $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ are power-controlled nodes, with power P_1, P_2, and P_3, respectively. The network edges, representing the RL transmission lines, are $\mathbf{c} = \{c_1, c_2, ..., c_5\}$, with each c_i having an associated pair of parameters (r_i, L_i). If we assign arbitrary directions to the edges of the graph, then we can define an incidence matrix $\mathcal{B} = \text{stack}\,(\mathcal{B}_{\mathcal{V}}, \mathcal{B}_{\mathcal{P}})$, where

$$\mathcal{B}_{\mathcal{V}} = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \end{bmatrix},$$
$$\mathcal{B}_{\mathcal{P}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \end{bmatrix}.$$

Then, the elements of the algebraic system (16), which is codified by $f(x) = 0$, with $x = \bar{V}$, are given by

$$A = \begin{bmatrix} \gamma_1 + \frac{1}{r_3} + \frac{1}{r_5} & 0 & -\frac{1}{r_5} \\ 0 & \gamma_2 + \frac{1}{r_1} + \frac{1}{r_4} & -\frac{1}{r_4} \\ -\frac{1}{r_5} & -\frac{1}{r_4} & \gamma_3 + \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{r_5} \end{bmatrix},$$
$$b = \text{stack}(P_i), \;\; w = \text{stack}\left(\frac{E}{r_3}, \frac{E}{r_1}, \frac{E}{r_2}\right),$$

where r_i and γ_i are the diagonal elements of the matrices R and G, respectively.

Taking the numerical values shown in Tables V and IV, we compute—through Lemma 3.1—an initial condition $x_0 \in \mathcal{E}$ given by

$$x_0 = 10^5 \cdot \text{stack}(6.66, 4.66, 5.99).$$

The particular solution $x(t, x_0)$ of $\dot{x} = f(x)$ is shown in Fig. 5. Clearly, none of its components converges to zero. Then,

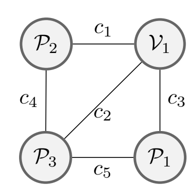

Fig. 4: Associated graph for the HVDC network studied in [22, Section V].

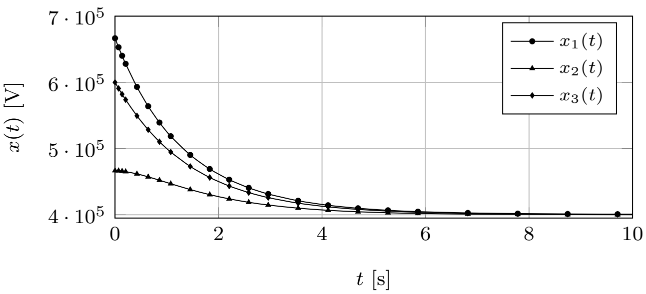

Fig. 5: Distinguished solution $x(t, x_0)$ converging to an equilibrium point. Once again, from Proposition 2.3 we establish that $x(t, x_0) \to \bar{x}_{\max}$ as $t \to \infty$.

TABLE V: Numerical parameters associated with the nodes for the network in Fig. 4.

Power converter	$\mathcal{V}_1$	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$
$V_{\mathcal{V}}^{(i)}$ (kV)	400	-	-	-
P_i (MW)	-	-160	140	-180
γ_i (µS)	-	0.02290	0.02290	0.3435

by Proposition 2.3, we establish that the limit of this solution is the dominant equilibrium point, $\bar{x}_{\max}$, of the system; its numerical value is given by

$$\bar{x}_{\max} = 10^5 \cdot \text{stack}(4.0054, 3.9991, 4.0043).$$

VI. Conclusions

A systematic methodology to analyze the behavior of the ODE (3) is presented in the paper. Exploiting the fact that it is a monotone dynamical system, we have described all possible scenarios for existence of its equilibria and, under minor extra assumptions, for their stability and uniqueness. It was proven that if equilibria exist, then, there is a distinguished one, denoted by $\bar{x}_{\max}$, which dominates—component-wise—all the other ones and attracts all the ODE trajectories starting from a well-defined domain. We have further provided an algorithm to establish whether solutions of the ODE will converge to $\bar{x}_{\max}$ or not and shown that the procedure is applicable even in the case when the system parameters are only known to live in a polytope.

These results have been applied to study the voltage-regularity of "lightly" loaded AC power systems and to give conditions for existence of equilibria in DC microgrids and MT-HVDC networks—all of them containing CPLs.

Finally, we have demonstrated via supporting numerical experiments on two benchmark power system models that our methodology performs very satisfactorily for realistic power system parameterizations.

Current research is under way in three directions. One is the incorporation in our analysis of voltage dynamics, either considering the swing equation or the loads' dynamics. Furthermore, we are also investigating diverse ODEs' integration techniques to identify efficient ways of implementing the numerical algorithm of Subsection III-A for large-scale systems. Finally, we are analyzing the possibility of dropping the condition (6) from our analysis and to detail the asymptotic behavior of the system (3) in this case.

Appendix A
Proof of Remark 2.1

Suppose that a system (2) meets Assumption 2.1 except for the claim $b_i \neq 0 \; \forall i$. We are going to show that this system can be reduced to an equivalent system (2) of a lower order that satisfies Assumption 2.1 in full, including the claim $b_i \neq 0 \; \forall i$.

To this end, it suffices to eliminate every variable x_k with $b_k = 0$ by using (1): $x_k = a_{kk}^{-1}[w_k - \sum_{j \neq k} a_{kj} x_j]$. This shapes any remaining equation (1) (with $i \neq k$) into

$$\sum_{j \neq k} \big[a_{ij} - a_{ik}a_{kj}/a_{kk}\big] x_j + b_i/x_i = w_i - w_k a_{ik}/a_{kk} \quad i \neq k.$$

In the l.h.S, the matrix A_r of the linear part is symmetric and its off-diagonal elements $a_{ij} - a_{ik}a_{kj}/a_{kk} \leq 0 \; \forall i \neq j, i, j \neq k$ since $a_{kk} > 0$ for the positive definite matrix A. Meanwhile, a symmetric matrix $\mathcal{A}$ is positive definite if and only if $-\mathcal{A}$ is Hurwitz. By [26, Th. 9.5], this is also equivalent to the fact that the equation $\mathcal{A}z = b$ has a unique root $z \geq 0$ whenever $b \geq 0$. The equation $A_r z = b$ is equivalent to the system $\sum_{j=1}^{n} a_{ij} x_j = b_i$ if $i \neq k$ and 0 otherwise. Since this system has a unique nonnegative solution x, and dropping x_k in x results in the solution z, we see that A_r is positive definite.

Consecutively eliminating all x_i with $b_i = 0$, we obtain an equivalent system (2) that meets Assumption 2.1 in full. □□□

Appendix B
Proofs of Lemma 3.1 and a related claim

Lemma B.1: The following system of inequalities is feasible

$$Az > 0, \qquad z > 0. \tag{20}$$

Proof: Suppose that the system (20) is infeasible. Then two open convex cones $A\mathcal{K}_+^n$ and $\mathcal{K}_+^n$ are disjoint and so can be separated by a hyperplane: there exists

$$\tau \in \mathbb{R}^n, \qquad \tau \neq 0 \tag{21}$$

such that

$$\tau^\top x \geq 0 \quad \forall x \in \mathcal{K}_+^n, \qquad \tau^\top x \leq 0 \quad \forall x \in A\mathcal{K}_+^n.$$

By continuity argument, these inequalities extend on the closures of the concerned sets:

$$\tau^\top x \geq 0 \quad \forall x \in \overline{\mathcal{K}}_+^n = \{x : x_i \geq 0\},$$
$$\tau^\top x \leq 0 \quad \forall x \in \overline{A\mathcal{K}_+^n} \supset A\overline{\mathcal{K}}_+^n.$$

Here the first relation implies that $\tau \in \overline{\mathcal{K}}_+^n \Rightarrow A\tau \in A\overline{\mathcal{K}}_+^n$ and so $\tau^\top A\tau \leq 0$ by the second one. Since A is positive definite by Assumption 2.1, the last inequality yields that $\tau = 0$, in violation of the second relation from (21). This contradiction completes the proof. □□□

Proof of Lemma 3.1. Let z be a solution of (20). It suffices to note that a solution of (7) can be built in the form $x := \mu z$ by picking $\mu > 0$ so that for all i,

$$\mu(Az)_i > \langle w_i \rangle + \frac{\langle -b_i \rangle}{\mu z_i} \Leftrightarrow \mu^2 (Az)_i - \mu \langle w_i \rangle - \frac{\langle -b_i \rangle}{z_i} > 0$$
$$\Leftrightarrow \mu > \frac{\langle w_i \rangle + \sqrt{\langle w_i \rangle^2 + 4(Az)_i \frac{\langle -b_i \rangle}{z_i}}}{2(Az)_i}.$$

APPENDIX C
TECHNICAL FACTS UNDERLYING PROPOSITIONS 2.1–2.3

In this section, we consider a C^1-map $g : \mathcal{K}_+^n \to \mathbb{R}^n$ and provide a general study of the ODE

$$\dot{x} = g(x), \qquad x \in \mathcal{K}_+^n, \tag{22}$$

under the following.

Assumption C.1: For any $x \in \mathcal{K}_+^n$, the off-diagonal elements of the Jacobian matrix $\nabla g(x)$ are nonnegative.

Assumption C.2: For any $x \in \mathcal{K}_+^n$, the Jacobian matrix $\nabla g(x)$ is symmetric.

For the convenience of the reader, we first recall some facts that will be instrumental in our study. The first group of them reflects that the system (22) is *monotone* (see [11] for a definition).

Proposition C.1: Let Assumption C.1 hold and let the order $\succ$ in $\mathbb{R}^n$ be either $\geq$ of $>$. For any solutions $x_1(t), x_2(t), x(t)$ of (22) defined on $[0,\tau], \tau > 0$, the following relations hold

$$x_2(0) \succ x_1(0) \Rightarrow x_2(t) \succ x_1(t) \; \forall t \in [0,\tau], \tag{23}$$

$$\begin{aligned} \dot{x}(0) \prec 0 \Rightarrow \dot{x}(t) \prec 0 \; \forall t \in [0,\tau], \\ \dot{x}(0) \succ 0 \Rightarrow \dot{x}(t) \succ 0 \; \forall t \in [0,\tau]; \end{aligned} \tag{24}$$

$$(x_+ > 0 \;\wedge\; g(x_+) > 0) \Rightarrow \quad \text{the domain} \quad \Upsilon_+ := \{x : x \succ x_+\} \text{ is positively invariant.} \tag{25}$$

Proof: Relation (23) is given by [11, Prop. 1.1 and Rem. 1.1, Ch. 3], whereas (24) is due to [11, Prop. 2.1, Ch. 3]. To prove (25), we consider the maximal solution $x_\dagger(t), t \in [0,\theta)$ of (22) starting from $x_\dagger(0) = x_+$. Since $\dot{x}_\dagger(0) = g(x_+)>0$, (24) guarantees that $x_\dagger(\cdot)$ constantly increases $\dot{x}_\dagger(t)>0 \; \forall t \in [0,\theta)$ and so $x_\dagger(t)>x_+ \; \forall t \in (0,\theta)$. Now let a solution $x(t), t \in [0,\tau], \tau \in (0,\infty)$ start in Υ_+. Then $x(0) \succ x_\dagger(0)$ and by (23), $x(t) \succ x_\dagger(t)>x_+$ if $t > 0$. So $x(t) \in \Upsilon_+$ for any $t \in [0,\tau] \cap [0,\theta)$. It remains to show that $\tau < \theta$ if $\theta < \infty$.

Suppose to the contrary that $\tau \geq \theta$. Letting $t \to \theta-$, we see that $\|x_\dagger(t)\| \to \infty$ by [10, Th. 3.1, Ch. II] since $x_\dagger(t) \succ x_+ > 0$. So $x(t) \succ x_\dagger(t) \Rightarrow \|x(t)\| \to \infty$. However, $\|x(t)\| \to \|x(\tau)\| < \infty$. This contradiction completes the proof. □□□

Let $x(t,a), t \in [0,\tau_a)$ stand for the maximal solution of (22) that starts at $t = 0$ with $a > 0$. The distance $\inf_{x' \in A} \|x - x'\|$ from point $x \in \mathbb{R}^n$ to a set $A \subset \mathbb{R}^n$ is denoted by $\mathbf{dist}(x, A)$.

Corollary C.1: Whenever $0 < a_1 \leq a \leq a_2$, we have $\tau_a \geq \min\{\tau_{a_1}, \tau_{a_2}\}$.

The following lemma is a trivial corollary of [26, Th. 9.5].[7]

Lemma C.1: A nonsingular matrix $A = A^\top$ with nonnegative off-diagonal elements is Hurwitz if

$$Ah > 0 \Rightarrow h \leq 0. \tag{26}$$

Lemma C.2: Let Assumptions C.1 and C.2 hold. Suppose that a solution $x(t), t \in [0,\infty)$ of (22) decays $\dot{x}(t) < 0 \; \forall t$ and converges to $\bar{x} > 0$ as $t \to \infty$. Then $\bar{x}$ is an equilibrium of the ODE (22). If this equilibrium is hyperbolic, it is locally asymptotically stable.

Proof: The first claim is given by [11, Prop. 2.1, Ch. 3]. By Lemma C.1, it suffices to show that $A := \nabla g(\bar{x})$ meets (26) to prove the second claim. Suppose to the contrary that $Ah > 0$ and $h_i > 0$ for some i and $h \in \mathbb{R}^n$. For $x_\varepsilon^0 := \bar{x} + \varepsilon h$ and small enough $\varepsilon > 0$, we have $g(x_\varepsilon^0) = g(\bar{x}) + \varepsilon Ah + o(\varepsilon) = \varepsilon Ah + o(\varepsilon) > 0, x_\varepsilon^0 > 0$, $x_{\varepsilon,i}^0 > \bar{x}_i$, and $x(0) \in \Upsilon_+ = \{x : x > x_\varepsilon^0\}$. Since the set Υ_+ is positively invariant by (25), $x(t) \in \Upsilon_+ \Rightarrow x_i(t) > x_{\varepsilon,i} > \bar{x}_i$, in violation of $x(t) \to \bar{x}$ as $t \to \infty$. This contradiction completes the proof. □□□

For any $x' \leq x'' \in \mathbb{R}^n$, we denote $\lceil x', x'' \rceil := \{x \in \mathbb{R}^n : x' \leq x \leq x''\}$.

Lemma C.3: Suppose that Assumption C.1 holds and $\bar{x} > 0$ is a locally asymptotically stable equilibrium. Its domain of attraction $\mathcal{A}(\bar{x}) \subset \mathcal{K}_+^n$ is open and

$$a', a'' \in \mathcal{A}(\bar{x}) \wedge a' \leq a'' \Rightarrow \lceil a', a'' \rceil \subset \mathcal{A}(\bar{x}). \tag{27}$$

Proof: Let $B(r,x)$ stand for the open ball with a radius of $r > 0$ centered at x. For any $a \in \mathcal{A}(\bar{x})$, we have $\tau_a = \infty$ and $x(t,a) \to \bar{x}$ as $t \to \infty$, whereas $B(2\varepsilon, \bar{x}) \subset \mathcal{A}(\bar{x})$ for a sufficiently small $\varepsilon > 0$ since $\bar{x}$ is locally asymptotically stable. Clearly, there is $\theta > 0$ such that $x(\theta, a) \in B(\varepsilon, \bar{x})$. By [10, Th. 2.1, Ch. V], there exists $\delta > 0$ such that whenever $\|a^\dagger - a\| < \delta$, the solution $x(\cdot, a^\dagger)$ is defined at least on $[0,\theta]$ and $\|x(\theta, a^\dagger) - x(\theta, a)\| < \varepsilon$. It follows that $x(\theta, a^\dagger) \in B(2\varepsilon, \bar{x})$ and so $x(\cdot, a^\dagger)$ is in fact defined on $[0,\infty)$ and converges to $\bar{x}$ as $t \to \infty$. Thus we see that $\|a^\dagger - a\| < \delta \Rightarrow a^\dagger \in \mathcal{A}(\bar{x})$, *i.e.*, the set $\mathcal{A}(\bar{x})$ is open.

Let $a \in \lceil a', a'' \rceil$. By Corollary C.1 and (23), $\tau_a = \infty$ and $x(t,a') \leq x(t,a) \leq x(t,a'') \; \forall t \geq 0$. Letting $t \to \infty$ shows that $x(t,a) \to \bar{x}$ and so $a \in \mathcal{A}(\bar{x})$. □□□

Lemma C.4: Let $\overline{x}_- \leq \overline{x}_+$ and let $D \subset \Xi := \lceil \overline{x}_-, \overline{x}_+ \rceil$ be an open (in Ξ) set such that *(i)* $\lceil x', x'' \rceil \subset D \; \forall x', x'' \in D$; *(ii)* either $\overline{x}_- \in D$ or $\overline{x}_+ \in D$; *(iii)* $D \neq \Xi$. Then there exists a continuous map $M : \Xi \to \Xi$ such that $M[\Xi] \subset \Xi_- := \Xi \backslash D$ and $M[x] = x \quad \forall x \in \Xi_-$.[8]

Proof: Let $\overline{x}_+ \in D$ for the definiteness; then $\overline{x}_- \notin D$ by (i) and (iii). It can be evidently assumed that $0 = \overline{x}_- < \overline{x}_+$. We denote $\chi_x(\theta) := \max\{x - \theta\zeta; 0\}$, where $\zeta := \text{stack}(1, \ldots, 1)$ and the $\max$ is meant component-wise. Evidently, $\Theta(x) := \{\theta \geq 0 : \chi_x(\theta) \in D\} = [0, \tau(x)) \; x \in D$, where $0 < \tau(x) < \infty$. For $x \notin D$, we put $\tau(x) := 0$. We are going to show first that the function $\tau(\cdot)$ is continuous on Ξ. To this end, it suffices to prove that $\tau(\bar{x}) = \tau_*$ whenever

$$\bar{x} = \lim_{k\to\infty} x^k, \quad x^k \in \Xi, \quad \text{and} \quad \tau_* = \lim_{k\to\infty} \tau(x^k).$$

Passing to a subsequence ensures that either $x^k \notin D \; \forall k$ or $x^k \in D \; \forall k$. In the first case, $\bar{x} \notin D$ since D is open. Then $\tau(\bar{x}) = 0 = \tau(x^k) = \tau_*$. Let $x^k \in D \; \forall k$. Since $\chi_{x^k}[\tau(x^k)] \notin D$ and D is open, letting $k \to \infty$ yields $\chi_{\bar{x}}[\tau_*] \notin D \Rightarrow \tau(\bar{x}) \leq \tau_*$. So the claim holds if $\tau_* = 0$. If $\tau_* > 0$, we pick $0 < \theta < \tau_*$. Then $\theta < \tau(x^k)$ for $k \approx \infty$, *i.e.*, $\chi_{x^k}(\theta) \in D$. Let x_i^ν be the ith component of $x^\nu \in \mathbb{R}^p$. Then

$$\tau_k' := \max\left\{\tau \geq 0 : \chi_{\bar{x}}(\tau) \geq \chi_{x^k}(\theta)\right\} = \max_{i : x_i^k \geq \theta} \left[\bar{x}_i - x_i^k + \theta\right].$$

Here the second $\max$ is over a nonempty set since $\chi_{x^k}(\theta) \in D \not\ni 0$. Thus $\tau_k' \to \theta$ as $k \to \infty$. By (i), $\chi_{\bar{x}}(\tau_k') \in D$ and so

[7] We thank an anonymous Rewiever for indicating this.

[8] In brief, this lemma says that Ξ_- is a retract of the convex set Ξ.

$\tau(\bar{x}) \geq \tau_k' \xRightarrow{k\to\infty} \tau(\bar{x}) \geq \theta \; \forall \theta < \tau_* \Rightarrow \tau(\bar{x}) \geq \tau_* \Rightarrow \tau(\bar{x}) = \tau_*$. Thus the function $\tau(\cdot)$ is continuous indeed. The needed map M is given by $M(x) := \chi_x[\tau(x)]$. □□□

Lemma C.5: Let Assumption C.1 hold and $0 < \overline{x}_- \leq \overline{x}_+, \overline{x}_- \neq \overline{x}_+$ be two locally asymptotically stable equilibria. Then there exists a third equilibrium $\bar{x}$ in between them $\overline{x}_- \leq \bar{x} \leq \overline{x}_+, \bar{x} \neq \overline{x}_-, \overline{x}_+$.

Proof: We denote $\Xi := [\overline{x}_-, \overline{x}_+]$, like in Lemma C.4. By Lemma C.3, the set $D_\pm := \mathcal{A}(\overline{x}_\pm) \cap \Xi$ meets the assumptions of Lemma C.4, which associates $D_\pm$ with a map $M_\pm$. Since the sets $D_\pm$ are open and disjoint, they do not cover the connected set Ξ. So the set $\Xi_\blacklozenge := \Xi \setminus (D_- \cup D_+)$ of all fixed points of the map $M = M_- \circ M_+$ is non-empty and compact. For all $a \in \Xi$, the solution $x(\cdot, a)$ is defined on $[0, \infty)$ by Corollary C.1 and $x(t, a) \in \Xi$ by (23). So the flow $\{\Phi_t(a) := x(t, a)\}_{t \geq 0}$ is well defined on Ξ, acts from Ξ into Ξ, and is continuous by [10, Th. 2.1, Ch. V]. The sets $D_\pm$ are positively and negatively invariant with respect to it:

$$\begin{aligned} a \in & D_\pm \Rightarrow \Phi_t(a) \in D_\pm \; \forall t \geq 0, \\ a \in & \Xi \wedge \left[\exists t \geq 0 : \Phi_t(a) \in D_\pm\right] \Rightarrow a \in D_\pm. \end{aligned}$$

It follows that $\Xi_\blacklozenge$ is positively invariant with respect to this flow. By the Brouwer fixed point theorem, the continuous map $\Phi_t \circ M : \Xi \to \Xi_\blacklozenge \subset \Xi$ has a fixed point $a_t = \Phi_t \circ M(a_t) \in \Xi$. Since $M(a_t) \in \Xi_\blacklozenge$ and $\Phi_t(\Xi_\blacklozenge) \subset \Xi_\blacklozenge$, we see that $a_t \in \Xi_\blacklozenge$ and so $M(a_t) = a_t$ and $a_t = \Phi_t(a_t)$.

Since $\Xi_\blacklozenge$ is compact, there exists a sequence $\{t_k > 0\}_{k=1}^\infty$ such that $t_k \to 0$ and $a_{t_k} \to \bar{x}$ as $k \to \infty$ for some point $\bar{x} \in \Xi_\blacklozenge$. Since $\overline{x}_\pm \notin \Xi_\blacklozenge$, we have $\bar{x} \neq \overline{x}_\pm$; meanwhile $\bar{x} \in \Xi_\blacklozenge \subset \Xi \Rightarrow \overline{x}_- \leq \bar{x} \leq \overline{x}_+$. Furthermore,

$$0 = \frac{\Phi_{t_k}(a_{t_k}) - a_{t_k}}{t_k} = t_k^{-1} \int_0^{t_k} g[x(t, a_{t_k})] \, dt \xrightarrow{k\to\infty} g(\bar{x}).$$

Thus we see that $g(\bar{x}) = 0$, *i.e.*, $\bar{x}$ is an equilibrium. □□□

Appendix D
Proofs of Propositions 2.1—2.3 and 3.1

We revert to study of the system (3) under Assumption 2.1.

Lemma D.1: Suppose that y belongs to the set (7). There exists $\theta \in (0, 1)$ such that the domain

$$\Xi_-(\theta) := \{x : 0 < x \leq \theta y\}$$

is globally absorbing, *i.e.*, the following statements hold:

(i) This domain is positively invariant: if a solution starts in $\Xi_-(\theta)$, it does not leave $\Xi_-(\theta)$;

(ii) Any solution defined on $[0, \infty)$ eventually enters $\Xi_-(\theta)$ and then never leaves this set.

Proof: Thanks to (7), there exists $\delta > 0$ such that

$$Ay > \text{stack}\left(\langle w_i \rangle + \frac{\langle -b_i \rangle}{y_i} + 3\delta\right). \tag{28}$$

We also pick $\theta \in (0, 1)$ so close to 1 that for any i, we have

$$[\theta - 1]\langle w_i \rangle + \delta\theta \geq 0, \quad [\theta - \theta^{-1}]\langle -b_i \rangle y_i^{-1} + \delta\theta \geq 0. \tag{29}$$

Let $x(\cdot)$ be a solution of (3). By the Danskin theorem [29], the function $\varrho(t) := \max_{i \in \{1,\ldots,n\}} x_i(t)/y_i$ is absolutely continuous and for almost all t, the following equation holds

$$\begin{aligned} \dot{\varrho}(t) &= \max_{i \in I(t)} \dot{x}_i(t)/y_i, \quad \text{where} \\ I(t) &:= \{i : x_i(t)/y_i = \varrho(t)\}. \end{aligned} \tag{30}$$

If $i \in I(t)$ and j, we have $x_i(t) = y_i \varrho(t), x_j(t) \leq y_j \varrho(t)$, and

$$\begin{aligned} \dot{x}_i(t) &\overset{(3)}{=} -a_{ii} x_i(t) + \sum_{j \neq i} \underbrace{[\; -a_{i,j} \;]}_{\geq 0 \text{ by Asm. 2.1}} x_j(t) - \frac{b_i}{x_i(t)} + w_i \\ &\leq -\varrho(t) \left(a_{ii} y_i + \sum_{j \neq i} a_{i,j} y_j\right) - \varrho(t)^{-1} \frac{b_i}{y_i} + w_i \\ &\overset{(28)}{\leq} -\varrho(t) \left[\langle w_i \rangle + \frac{\langle -b_i \rangle}{y_i} + 3\delta\right] + \varrho(t)^{-1} \frac{\langle -b_i \rangle}{y_i} + \langle w_i \rangle \\ &= -\delta \varrho(t) - \{[\varrho(t) - 1]\langle w_i \rangle + \delta \varrho(t)\} \\ &\quad - \left\{[\varrho(t) - \varrho(t)^{-1}] \frac{\langle -b_i \rangle}{y_i} + \delta \varrho(t)\right\}. \end{aligned} \tag{31}$$

Hence whenever $\varrho(t) \geq \theta \in (0, 1)$,

$$\begin{aligned} \dot{x}_i(t) \leq& -\delta \varrho(t) - \{[\theta - 1]\langle w_i \rangle_+ + \delta\theta\} \\ &- \left\{[\theta - \theta^{-1}] \frac{\langle b_i \rangle_-}{y_i} + \delta\theta\right\} \overset{(29)}{\leq} -\delta \varrho(t). \end{aligned}$$

So (30) implies that $\varrho(t) > \theta \Rightarrow \dot{\varrho}(t) \leq -\delta \varrho(t) \leq -\delta\theta$.[9] Claims (i) and (ii) are immediate from this entailment. □□□

Lemma D.2: Claim *II)* of Proposition 2.3 holds.

Proof: This is immediate from (24) since for any distinguished solution $x(\cdot)$ and $y := x(0)$,

$$\begin{aligned} \dot{x}(0) &\overset{(3)}{=} -Ay + \text{stack}\left(-\frac{b_i}{y_i} + w_i\right) \\ &\leq -Ay + \text{stack}\left(\frac{\langle -b_i \rangle}{y_i} + \langle w_i \rangle\right) \overset{(7)}{<} 0. \end{aligned}$$ □□□

Lemma D.3: Suppose that a solution $x(\cdot)$ of (3) cannot be extended from $[0, \tau)$ with $\tau < \infty$ to the right. Then there is i such that $b_i > 0$ and $x_i(t) \to 0, \dot{x}_i(t) \to -\infty$ as $t \to \tau_-$.

Proof: By Lemma 3.1, there is a solution $y > 0$ for (7). Via multiplying y by a large enough factor, we ensure that $y > x(0)$. Let $x_\uparrow(\cdot)$ be the distinguished solution starting with $x_\uparrow(0) = y$. By Lemma D.2, $x_\uparrow(t) \leq y$ for $t \geq 0$, and $x(t) \leq x_\uparrow(t)$ on the intersection of the domains of definitions of $x(\cdot)$ and $x_\uparrow$ by (23). Then by [10, Th. 3.1, Ch. II], $x(t)$ converges to the boundary of $\mathcal{K}_+^n$ as $t \to \tau_-$ and is bounded, *i.e.,*

$$\min_i x_i(t) \to 0 \text{ as } t \to \tau_-, \quad c := \sup_{t \in [0, \tau)} \|x(t)\| < \infty. \tag{32}$$

Putting $W := \max_i \left[|w_i| + c \sum_j |a_{ij}|\right]$, we see that

$$\begin{aligned} \dot{x}_i(t) &\overset{(31)}{=} -\sum_j a_{i,j} x_j(t) - \frac{b_i}{x_i(t)} + w_i \\ &\in \left[-W - \frac{b_i}{x_i(t)}, W - \frac{b_i}{x_i(t)}\right], \end{aligned}$$

[9] In fact, this holds for almost all t such that the premises are true.

$$b_i < 0 \wedge x_i(t) \leq \frac{|b_i|}{2W} \Rightarrow \dot{x}_i(t) \geq W > 0, \tag{33}$$

$$\begin{aligned} b_i > 0 \wedge x_i(t) \leq \frac{|b_i|}{2W} &\Rightarrow \dot{x}_i(t) \leq -\frac{b_i}{2x_i(t)} < 0 \\ &\Rightarrow x_i^2(\theta) \leq x_i^2(t) - b_i(\theta - t)\ \forall \theta \in [t, \tau). \end{aligned} \tag{34}$$

Due to (33), $x_i(t)$ is separated from zero if $b_i < 0$. So by (32), there exists i such that $b_i > 0$ and for any $\varepsilon > 0$, arbitrarily small left vicinity $(\tau_{-}\delta, \tau), \delta \approx 0$ of τ contains points t with $x_i(t) < \varepsilon$. Then for $\varepsilon < \frac{|b_i|}{2W}$, formula (34) guarantees that $x_i(t') < \varepsilon\ \forall t' \in (t, \tau)$. Overall, we see that $x_i(t) \to 0$ as $t \to \tau_{-}$; then $\dot{x}_i(t) \to -\infty$ as $t \to \tau_{-}$ by (34). □□□

Lemma D.4: Suppose that the condition (6) holds. *(i)* Stable equilibria of (3) (if exist) are locally asymptotically stable. *(ii)* Let $0 < x^{-} \leq x^0 \leq x^{+}$ be equilibria of (3). If $x^{\pm}$ are stable and all b_i's are of the same sign, x^0 is also stable.

Proof: By (6) and (3), the Jacobian matrix

$$\begin{aligned} \nabla f(x) = A(k) &:= -A + \mathbf{diag}\left[k_i\right], \\ k := k(x) &:= \text{stack}\left(b_i x_i^{-2}\right) \end{aligned} \tag{35}$$

has no eigenvalues with the zero real part at any equilibrium x. So for any stable equilibrium x, the matrix from (35) is Hurwitz and so x is locally asymptotically stable. Since $A(k)^\top = A(k)$ by Assumption 2.1, $A(k)$ is Hurwitz if and only if the following quadratic form in $h \in \mathbb{R}^n$ is negatively definite

$$Q_x(h) := -h^\top A h + \sum_{i=1}^{n} k_i(x) h_i^2.$$

Thus both forms $Q_{x^\pm}$ are negatively definite. Meanwhile, $k_i(x^0) \leq k_i(x^-) \forall i$ if $b_i > 0\ \forall i$, whereas $k_i(x^0) \leq k_i(x^+) \forall i$ if $b_i < 0\ \forall i$. In any case, Q_{x^0} is upper estimated by a negatively definite quadratic form (either Q_{x^-} or Q_{x^+}) and so is negatively definite as well. □□□

Corollary D.1: Suppose that the condition (6) holds, $0 < x^{(0)} \leq x^{(1)}$ are stable equilibria of (3) and all b_i's are of the same sign. Then $x^{(0)} = x^{(1)}$.

Proof: Suppose to the contrary that $x^{(0)} \neq x^{(1)}$. By Lemma C.5 and (i) of Lemma D.4, there exists one more equilibrium $x^{(1/2)}$ in between $x^{(0)}$ and $x^{(1)}$, *i.e.*, $x^{(0)} \leq x^{(1/2)} \leq x^{(1)}$ and $x^{(1/2)} \neq x^{(0)}, x^{(1)}$. By (ii) of Lemma D.4, this newcoming equilibrium $x^{(1/2)}$ is stable. This permits us to repeat the foregoing arguments first for $x^{(0)}$ and $x^{(1/2)}$ and second for $x^{(1/2)}$ and $x^{(1)}$. As a result, we see that there exist two more stable equilibria $x^{(1/4)} \in \lceil x^{(0)}, x^{(1/2)} \rceil$ and $x^{(3/4)} \in \lceil x^{(1/2)}, x^{(1)} \rceil$ that differ from all previously introduced equilibria. This permits us to repeat the foregoing arguments once more to show that there exist stable equilibria $x^{(1/8)}, x^{(3/8)}, x^{(5/8)}, x^{(7/8)}$ such that $x^{(i/8)} \leq x^{(j/8)}\ \forall 0 \leq i \leq j \leq 8$ and $x^{(i/8)} \neq x^{(j/8)}\ \forall 0 \leq i, j \leq 8, i \neq j$. By continuing likewise, we assign a stable equilibrium $x^{(r)}$ to any number $r \in [0, 1]$ whose representation in the base-2 numeral system is finite (*i.e.*, number representable in the form $r = j2^{-k}$ for some $k = 1, 2, \ldots$ and $j = 0, \ldots, 2^k$) and ensure that these equilibria are pairwise distinct and depend on r monotonically: $x^{(r)} \leq x^{(\varrho)}$ whenever $0 \leq r \leq \varrho \leq 1$.

Since all they lie in the compact set $\lceil x^{(0)}, x^{(1)} \rceil$, there is a sequence $\{r_k\}_{k=1}^{\infty}$ of pairwise distinct numbers r's for which $\exists \bar{x} = \lim_{k \to \infty} x^{(r_k)}$. Then $\bar{x} \in \lceil x^{(0)}, x^{(1)} \rceil$ and so $\bar{x} > 0$ and $f(\bar{x}) = \lim_{k\to\infty} f[x^{(r_k)}] = 0$, *i.e.*, $\bar{x}$ is an equilibrium. Then the Jacobian matrix $\nabla f(\bar{x})$ is nonsingular, as was remarked just after (35). However, this implies that in a sufficiently small vicinity V of $\bar{x}$, the equation $f(x) = 0$ has no roots apart from $\bar{x}$ in violation of $x^{(r_k)} \in V\ \forall k \approx \infty$ and $x^{(r_k)} \neq x^{(r_l)}\ \forall k \neq l$. The contradiction obtained completes the proof. □□□

Proof of Proposition 2.3: Claim I) is given by Lemma 3.1. *Claim II)* is justified by Lemma D.2.

Claim III) Let $x(t), t \in [0, t_f)$ be a distinguished solution. If $t_f < \infty$, then III.i) of Proposition 2.3 holds by Lemma D.3. Suppose that $t_f = \infty$. Then the limit $\bar{x}$ from (9) exists due to **II)**, and $\bar{x} \geq 0$. We are going to show that in fact $\bar{x} > 0$.

Suppose to the contrary that $\bar{x}_i = 0$ for some i. Then $x_i(t) \to 0$ as $t \to \infty$, (33) means that $b_i > 0$, and (34) (where $\tau = \infty$ now) implies that $\|x(\theta)\|^2$ assumes negative values for large enough θ. This assures that $\bar{x} > 0$ and so (9) does hold. By Lemma C.2, $\bar{x}$ is an equilibrium.

Now suppose that III.i) holds for a distinguished solution $x_\dagger(\cdot)$. Suppose that there is another distinguished solution $x(\cdot)$ for which III.i) is not true. Then $x(\cdot)$ is defined on $[0, \infty)$ by Lemma D.3 and also $\exists \bar{x} = \lim_{t\to\infty} x(t) > 0$ by the foregoing. By (ii) of Lemma D.1 (with $y := x_\dagger(0)$), $x(\sigma) \leq \theta x_\dagger(0) \leq x_\dagger(0)$ for large enough σ. By applying (23) to $x_1(t) := x(t + \sigma)$ and $x_2(t) = x_\dagger(t)$, we see that $x(t + \sigma) \leq x_\dagger(t)$ and so $x_i(t)$ goes to zero in a finite time, in violation of $\bar{x} > 0$ and *II)*. This contradiction proves that III.i) holds simultaneously for all distinguished solutions.

Since III.i) and III.ii) are mutually exclusive and complementary, we see that either III.i) holds for all distinguished solutions, or III.ii) holds for all of them.

Let III.ii) hold. As was shown in the paragraph prior to the previous paragraph, $x(t+\sigma) \leq x_\dagger(t)$ for any two distinguished solutions $x(\cdot)$ and $x_\dagger(\cdot)$. Hence $\lim_{t\to\infty} x(t) \leq \lim_{t\to\infty} x_\dagger(t)$. By flipping $x(\cdot)$ and $x_\dagger(\cdot)$ here, we see that these limits coincide, *i.e.,* the limit (9) is the same for all distinguished solutions.

Claim IV) follows from Lemmas D.1 and D.3 since any equilibrium is related to a constant solution defined on $[0, \infty)$.

Claim V) Suppose that III.ii) holds. Let $\bar{x}_{\mathtt{max}}$ stand for the limit (9). By (9) and Lemma C.2, $\bar{x}_{\mathtt{max}}$ is an equilibrium. Let us consider a solution $x(\cdot)$ defined on $[0, \infty)$ and a distinguished solution $x_\dagger(\cdot)$. By retracing the above arguments based on (ii) of Lemma D.1, we see that $x(\varsigma + t) \leq x_\dagger(t)\ \forall t \geq 0$ for some $\varsigma \geq 0$. By considering here a constant solution $x(\cdot)$ and letting $t \to \infty$, we see that $\bar{x}_{\max}$ dominates any other equilibrium.

Let $x(0) \geq \bar{x}_{\max}$. By (23), $x(t) \geq \bar{x}_{\max}$ on the domain Δ of definition of $x(\cdot)$ and so $\Delta = [0, \infty)$ by Lemma D.3. Thus we see that $x_{\max} \leq x(\varsigma + t) \leq x_\dagger(t)\ \forall t \geq 0$ for some $\varsigma \geq 0$. It follows that $x(t) \to x_{\max}$ as $t \to \infty$, *i.e.,* the equilibrium $x_{\max}$ is attractive from above by Definition 2.1. □□□

Proof of Proposition 2.1: This proposition is immediate from Proposition 2.3.

Proof of Proposition 2.2: We establish the proofs of the proposition's claims one by one.

Claim Y3) The set $\Pi := \{z \in \mathbb{R}^n : \det[A - \text{diag}(z_i)] = 0\}$ is closed and for any i and given z_j's with $j \neq i$, its section $\{z_i \in \mathbb{R} : \text{stack}(z_1, \ldots, z_n) \in \Pi\}$ has no more than n elements. So the measure of Π is zero by the Fubini theorem. The function $x \in \mathcal{K}_+^n \mapsto g(x) := \text{stack}(b_i x_i^{-2})$ diffeomorphically maps $\mathcal{K}_+^n$ onto an open subset of $\mathbb{R}^n$. Hence the inverse image $\Pi_\downarrow := g^{-1}(\Pi)$ is closed, has the zero measure and, due to these two properties, is nowhere dense.

Let C be the set of all critical points of the semi-algebraic map [27] $x \in \mathcal{K}_+^n \mapsto h(x) := Ax + \text{stack}(b_i x_i^{-2}) \in \mathbb{R}^n$, *i.e.*, points x such that the Jacobian matrix $\nabla h(x)$ is singular. By the extended Sard theorem [28], the set of critical values $h(C)$ has the zero measure and is nowhere dense. Meanwhile, the restriction $h_{\mathcal{K}_+^n \setminus C}$ is a local diffeomorphism and so the image $h(\Pi_\downarrow \setminus C)$ is nowhere dense and has the zero Lebesgue measure. It remains to note that the set of w's for which the condition (6) does not hold lies in $h(\Pi_\downarrow \setminus C) \cup h(C)$. □□□

Claim Y1) Let the condition (6) and *S2)* in Proposition 2.1 hold. Then $\bar{x}_{\max}$ is locally asymptotically stable by Lemma C.2 and *II), V)* in Proposition 2.3. It remains to show that there exist only finitely many equilibria $\bar{x}^k$.

Suppose the contrary. Since all equilibria lie in the compact set $\{x : 0 \leq x \leq \bar{x}_{\max}\}$, there exists an infinite sequence $\{\bar{x}^{k_s}\}_{s=1}^\infty$ of pairwise different equilibria that converges $\bar{x}^{k_s} \to \bar{x}$ as $t \to \infty$ to a point $\bar{x} \geq 0$. The estimates (33), (34) applied to any equilibrium solution $x(\cdot)$ assure that $x_i \geq |b_i|/(2W)$ on it, where $W := \max_i [|w_i| + c \sum_j |a_{ij}|]$ and c is any upper bound on $\|x(t)\|$. For the solutions related to the convergent and so bounded sequence $\{\bar{x}^{k_s}\}_{s=1}^\infty$, this bound can be chosen common. As a result, we infer that $\bar{x} > 0$ and so $f(\bar{x}) = \lim_{s\to\infty} f[x^{k_s}] = 0$, *i.e.*, $\bar{x}$ is an equilibrium. Then the Jacobian matrix $\nabla f(\bar{x})$ is nonsingular, as was remarked just after (35). This implies that in a sufficiently small vicinity V of $\bar{x}$, the equation $f(x) = 0$ has no roots apart from $\bar{x}$, in violation of $x^{k_s} \in V \ \forall s \approx \infty$ and $x^{k_s} \neq x^{k_r} \ \forall s \neq r$. This contradiction completes the proof.

Claim Y2): The proof of this claim is established by Corollary D.1.

□□□

Proof of Proposition 3.1: Let $f^-(\cdot), f(\cdot), f^+(\cdot)$ be defined by (3) for $\mathfrak{C}^-, \mathfrak{C}, \mathfrak{C}^+$, respectively. Retracing the proof of Lemma 3.1 demonstrates existence of $a \in \mathcal{K}_+^n$ such that $A^+ a > \text{stack}\big(\max\{\langle w_i^-\rangle; \langle w_i\rangle\langle w_i^+\rangle\} + \frac{\max\{\langle -b_i^-\rangle; \langle -b_i\rangle\langle -b_i^+\rangle\}}{a_i}\big)$. By Definition 2.2, the solutions of $\dot{x} = f(x)$ and $\dot{x}^\pm = f^\pm(x^\pm)$ that start with a are distinguished for the respective ODE's (3). Here $f^-(x) \leq f(x) \leq f^+(x) \ \forall x \in \mathcal{K}_+^n$ due to (11). So by [30, Th. 8.1, Ch. II],

$$x^-(t) \leq x(t) \leq x^+(t),$$

where each inequality holds whenever its both sides are defined for the concerned t. So *i)–iii)* are immediate from *III–V)* in Proposition 2.3.

Now let $A^+ \to A^-, b_i^+ \to b_i^-, w^+ \to w^-$. For the left hand sides of these relations, there exist respective constant upper bounds $\hat{A}^+, \hat{b}_i^+, \hat{w}^+$ that satisfy Assumption 2.1. Let $\hat{x}_{\max}$ be the dominant equilibrium related to these bounds. By (12), $x_{\max}^- \leq x_{\max}^+ \leq \hat{x}_{\max}^+$ and so the variety of $x_{\max}^+$'s is bounded. For any limit point $\bar{x}$ of $x_{\max}^+$'s, we have $0 < x_{\max}^- \leq \bar{x}$ and $\bar{x}$ is the equilibrium of the $\mathfrak{C}^-$-related system by the continuity argument. So $x_{\max}^- = \bar{x}$ by the definition of the dominant equilibrium. Thus all limit points of $x_{\max}^+$'s are the same and equal $x_{\max}^-$. Hence $x_{\max}^+ \to x_{\max}^-$. □□□

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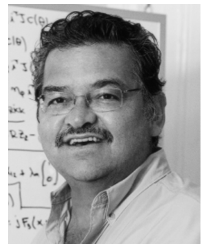

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