Modelling Start-Up Costs of Multiple Technologies in Electricity Markets

by

Ludwig Kuntz, Felix Müsgens

Abstract:

This paper analyzes the effects of start-up costs of different technologies in providing electricity power. We explicitly solve a simplified linear formulation of the dispatch problem. Transforming this primal problem, we show that dominated technologies should be used only in the case of limited availability of efficient technologies. Furthermore, we develop a 'pen and paper' algorithm to determine the optimal dispatch.

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Author Affiliations:

Ludwig Kuntz
Department of Economics
University of Köln
Albertus-Magnus-Platz
50923 Cologne
Germany

Phone: +49 /(0)221/4705416
Fax : +49 /(0)221/ 4705418
email: kuntz@wiso.uni-koeln.de

Felix Müsgens
Department of Economics
University of Köln
Albertus-Magnus-Platz
50923 Cologne
Germany

Phone: +49 /(0)221/4705490
Fax : +49 /(0)221/ 4705485
email: felix.muesgens@uni-koeln.de

I. INTRODUCTION

Starting up a power plant causes costs even before the actual production of electricity starts. The power plant's boiler has to be preheated and the plant has to be synchronized with the electricity network. Furthermore, attrition increases significantly due to the high variations in temperature during the start-up. These costs can be significant; if a hard coal plant is started up every day and produces during twelve peak hours (hours with high demand), the share of start-up costs for this plant is approximately 15% of total generation costs.[1] Hence, start-up costs are an important parameter in the optimization of electricity markets.

Neglecting start-up costs, the problem of determining an efficient plant dispatch (production schedule) to serve any given load profile is trivial. In this case, the technology with lowest variable generation costs is chosen for production as long as capacity of that technology is available. Start-up costs complicate matters. As they are independent of the following length of production, they add a fixed cost component. The resulting optimization problem is called a unit commitment problem. Unit commitment problems are often analyzed in the context of mixed integer (MIP) models (e.g. Bard 1988). While this approach is the most exact for the problem at hand, it has some disadvantages. Besides being computationally demanding, the interpretation of the dual variables in these mixed integer problems is difficult. Both O'Neill et al. (2005) and Hogan and Ring (2003) describe this problem in great detail. They also present a solution which is to solve the MIP problem first and then feed the solution to the integer variables of the MIP problem as constraints into a linear model.

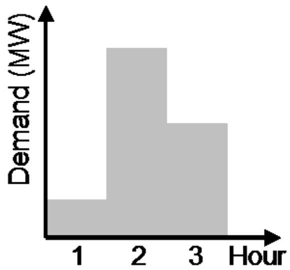
We follow the simpler approach taken by most practitioners and analyze a linear model directly. For this model, we will develop a simple algorithm to determine the optimal solution of the problem. The algorithm answers the question of which technology will be started up in which period and for how long it will operate in the cost-minimizing optimal solution. The algorithm works for any (positive) demand profile. However, we need some further simplifying assumptions to develop the algorithm and determine the cost minimizing solution. In particular, we abstract from capacity limitations, partial load operation and hydro storage capacity. Furthermore, we assume an inelastic short term demand for electricity.

In the remaining part of the introduction, we will present a simple example to highlight the effect of start-up costs on a cost minimizing dispatch. The example also demonstrates the intuition of the formal algorithm developed in the main part of the paper. In this example, we assume an exogenous electricity demand of one, five, and three MW for a three hour time period (left part of Figure 1). There are three different electricity generation technologies ('Tech'), 'a', 'b', and 'c'. We show the technologies' different start-up costs ('SUC', Euro/MW) and different variable generation costs ('VC', Euro/MWh) in the middle of the figure. For the load profile in this example, it is rather obvious to determine the optimal dispatch (right part of the figure): One MW of technology 'c' is started up in the first hour (producing for all three hours). Two

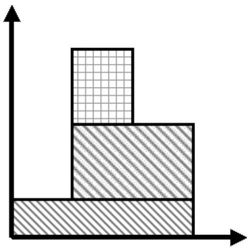
[1] This result holds for assumed variable generation costs of 20 Euro/MWh and start-up costs of 45 Euro/MW.

additional MW of technology ‘c’ are started in the second hour, producing during hours 2 and 3. Furthermore, two MW of technology ‘a’ are started in hour 2, producing in that hour only. Note that technology ‘b’ is never producing, as either technology ‘a’ or technology ‘c’ is cheaper for all possible durations of production.

Figure 1: An Example for Demand, Production Costs, and Plant Dispatch



Tech	SUC	VC	
a	10	30	
b	25	25	
c	30	15	



The paper is structured as follows: section II presents the model used to derive our results. Section III presents the results. The latter section includes the formal solution of the previous example derived with the algorithm. Section IV concludes the paper.

II. MODEL

We introduce the following linear optimization model to determine the cost minimizing power plant dispatch. The objective function is the minimization of total costs (TC):

(1) Minimize with respect to x_{ij}, x_{ij}^{+}, and x_{ij}^{-}: $TC = \sum_{j=1}^{n}\sum_{i=1}^{s}\left(x_{ij}\cdot vc_i + x_{ij}^{+}\cdot sc_i\right)$.

The model distinguishes three different groups of variables. x_{ij} is the production of technology i in period j, x_{ij}^{+} is the amount of capacity newly started and x_{ij}^{-} the amount of capacity shut down, $i \in S = \{1,\ldots,s\}$, $j \in N = \{1,\ldots,n\}$. Note that x_{ij}^{-} does not appear in the objective function because the shut down of capacity does not inflict any costs. Costs incurred by producing or by starting up a unit of x_{ij} are vc_i (variable generation costs) and sc_i (start-up costs), respectively.

Total costs are minimized subject to the following constraints:

(2) $$d_j - \sum_{i=1}^{s} x_{ij} = 0 \qquad j \in N$$

(3) $$x_{ij} - \sum_{k=1}^{j}\left(x_{ik}^{+} - x_{ik}^{-}\right) \le 0 \qquad i \in S, j \in N$$

(4) $$\alpha_i \cdot \left(\sum_{k=1}^{j}\left(x_{ik}^{+} - x_{ik}^{-}\right)\right) - x_{ij} \le 0 \qquad i \in S, j \in N$$

(5) $$\sum_{k=1}^{j}\left(x_{ik}^{+}-x_{ik}^{-}\right)-\overline{x}_{i}\leq 0 \qquad i\in S,\, j\in N$$

(6) $$x_{ij},x_{ij}^{+},x_{ij}^{-}\geq 0 \qquad i\in S,\, j\in N$$

Constraint (2) states that aggregated production equals demand d in each period. (3) assures that only capacity previously started can produce and (4) is a partial load constraint guaranteeing that at least a share α_s of previously started capacity is used for production. This partial load constraint is due to technical limitations on the operation of power plants. Constraint (5) states that total capacity started up and ready to produce cannot exceed the installed available capacity $\overline{x}_i$. Constraint (6) ensures that all variables are positive.

For the analysis in this paper, however, we assume that there is so much capacity that (5) is never binding. Furthermore, we set $\alpha_i = 1 \;\forall i\in S$ which means ignoring partial load operation. In that case, inequalities (3) and (4) simplify to one equation:

(7) $$x_{ij}-\sum_{k=1}^{j}\left(x_{ik}^{+}-x_{ik}^{-}\right)=0 \qquad i\in S,\, j\in N$$

We formulated that simplified problem (i.e. (1), (2), (6), and (7)) in vector notation as a maximization problem. Then, we face the following **primal** problem:

(PP) $$\begin{aligned}&\max\left\langle c,x\right\rangle\\ &\tilde{A}x=\tilde{d}\\ &x\geq 0\end{aligned}$$

where

$$x=(x_{11},\ldots,x_{1n},\ldots,x_{s1},\ldots,x_{sn},x_{11}^{+},x_{11}^{-},\ldots,x_{1n}^{+},x_{1n}^{-},\ldots,x_{s1}^{+},\ldots,x_{sn}^{-})^{T}\in\mathbb{R}^{3sn},$$

$$c=(-vc_{1},\ldots,-vc_{1},\ldots,-vc_{s},\ldots,-vc_{s},-sc_{1},0,\ldots,-sc_{1},0,\ldots,-sc_{s},0,\ldots,-sc_{s},0)^{T}\in\mathbb{R}^{3sn},$$

$\tilde{d}=(d_{1},\ldots,d_{n},0,\ldots,0)^{T}\in\mathbb{R}^{+(s+1)n}$, and

$$\tilde{A}=\begin{pmatrix} Id^{n} & Id^{n} & Id^{n} & \ldots & Id^{n} & 0 & 0 & 0 & \ldots & 0\\ Id^{n} & 0 & 0 & & 0 & B & 0 & 0 & & 0\\ 0 & Id^{n} & 0 & & 0 & 0 & B & 0 & & 0\\ 0 & 0 & Id^{n} & & 0 & 0 & 0 & B & & 0\\ \ldots & & & \ldots & & & & & \ldots & \\ 0 & 0 & 0 & & Id^{n} & 0 & 0 & 0 & & B \end{pmatrix}\in\mathbb{R}^{(s+1)n}\times\mathbb{R}^{3sn},$$

$$B=\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0\\ -1 & 1 & -1 & 1 & 0 & & 0 & 0\\ \vdots & & & & & \ddots & & \\ -1 & 1 & -1 & 1 & -1 & & -1 & 1 \end{pmatrix}\in\mathbb{R}^{n}\times\mathbb{R}^{2n}.$$

Id^n is the identity matrix with dimension n.

Transforming (PP) into its **dual** program (cf. e.g. Schrijver 1998) and slightly modification gives the following program:

$$\text{(DP):}\quad \begin{aligned} &\min \langle d, \lambda_0 \rangle \\ &\lambda_0 \geq -\lambda_i - vc_i 1_n \quad i \in S \\ &0 \leq A\lambda_i \leq sc_i 1_n \quad i \in S \end{aligned}$$

with

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & & \\ & & \dots & \\ 0 & & & 1 \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^n,\ d = (d_1, \dots, d_n)^T,\ \lambda_i = (\lambda_{i1}, \dots, \lambda_{in})^T,\ i \in S \cup \{0\}, \text{ and}$$

$$1_n = (\underbrace{1, \dots., 1}_{n})^T.$$

For the rearrangement we used that $B^T \lambda_i \geq (\underbrace{-sc_i, 0, \dots, -sc_i, 0}_{2n})^T$ for $i \in S$ if and only if

$0 \leq A\lambda_i \leq sc_i \cdot 1_n \;\; i \in S$.

III. RESULTS

Time-Dependent Generation Costs and Dominated Technologies

Theorem 1:

(DP) is equivalent to the following linear program:

$$\text{(DP*)}\quad \begin{aligned} &-\max < d, \tilde{\lambda}_0 > \\ &A_j \tilde{\lambda}_0 \leq \gamma_j \cdot 1_{n-j+1}, \text{for } j \in N \end{aligned},$$

where $\tilde{\lambda}_0 = -\lambda_0$, $A_j = \begin{pmatrix} \overbrace{\overbrace{1 \;\; 1 \;\; \cdots \;\; 1}^{j} \;\; 0 \;\; 0 \;\; \cdots \;\; 0 \;\; 0 \;\; 0 \;\; \cdots \;\; 0 \;\; 0}^{n} \\ 0 \;\; 1 \;\; \;\; 1 \;\; 1 \;\; 0 \;\; \;\; 0 \;\; 0 \;\; 0 \;\; \;\; 0 \;\; 0 \\ \vdots \qquad\qquad \ddots \qquad \vdots \qquad \ddots \\ 0 \;\; 0 \;\; \;\; 0 \;\; 0 \;\; 0 \;\; \;\; 0 \;\; 1 \;\; 1 \;\; \;\; 1 \;\; 0 \\ 0 \;\; 0 \;\; \cdots \;\; 0 \;\; 0 \;\; 0 \;\; \cdots \;\; 0 \;\; 0 \;\; \underbrace{1 \;\; \cdots \;\; 1 \;\; 1}_{j} \end{pmatrix} \in \mathbb{R}^{n-j+1} \times \mathbb{R}^n$

for $j < \dfrac{n}{2}$ and

$$A_j = \begin{pmatrix} \overbrace{1 \quad 1 \quad \cdots \quad 1 \quad 1 \quad 1 \quad \cdots \quad 1}^{j} & \overbrace{0 \quad 0 \quad \cdots \quad 0 \quad 0}^{n-j} \\ 0 \quad 1 \quad \quad 1 \quad 1 \quad 1 \quad \quad 1 & 1 \quad 0 \quad \quad 0 \quad 0 \\ \vdots \quad \quad \ddots & \vdots \\ 0 \quad 0 \quad \quad 0 \quad 1 \quad 1 \quad \quad 1 & 1 \quad 1 \quad \quad 1 \quad 0 \\ 0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad 1 \quad \cdots \quad 1 & 1 \quad 1 \quad \cdots \quad 1 \quad 1 \end{pmatrix} \in \mathbb{R}^{n-j+1} \times \mathbb{R}^{n} \text{ for } j \geq \frac{n}{2}.$$

Furthermore, $\gamma_j = \min_{i \in S} \{sc_i + j \cdot vc_i\}$.

Thus, γ_j is the cost minimum across all technologies for serving a flat demand of one unit for j hours. These costs are the sum of start-up costs and *j-times* variable generation costs. The optimal solution to (DP*) is only restricted by γ_j. Hence, other facts, for example the total number of different generation technologies, do not enter the problem. In addition, if a technology k is dominated by another technology i, i.e. $sc_i + j \cdot vc_i \leq sc_k + j \cdot vc_k$ for every $j \in N$, the dominated technology is never used for production and can be removed from the problem without changing the optimal solution. This follows because dominated technologies do not appear in the γ_j. However, this is only true when there is sufficient capacity of the dominating technologies available.

Algorithm to Determine the Optimal Plant Dispatch

Firstly, we define a_{ij} such that $\left(a_{1j}^T, \ldots, a_{n-1+j\ j}^T\right) := A_j$, $j \in N$ and $a_{i0} = \left(\underbrace{0,...,0}_{n}\right)^T$. Secondly, we construct a modified primal problem (PP*) which is the dual problem of (DP*).

$$\text{(PP*)} \quad -\min \sum_{j=1}^{n} \sum_{i=1}^{n-j+1} \tilde{x}_{ij} \cdot \gamma_j$$

$$\text{s.t. } \sum_{j=1}^{n} \sum_{i=1}^{n-j+1} \tilde{x}_{ij} \cdot a_{ij} = d \text{ and } \tilde{x}_{ij} \geq 0 \text{ for } j \in N,\ i = 1, ..., n - j + 1$$

The $\tilde{x}_{ij}$ in (PP*) have a straight forward interpretation. $\tilde{x}_{ij}$ is the amount of capacity that starts producing in hour i and produces for the next consecutive j hours. Based on this modified primal problem, we developed the following algorithm to identify the optimal production.

Theorem 2:

Suppose $d \geq 0$. An optimal solution $\tilde{x}^* = (\tilde{x}_{1n}^*, ..., \tilde{x}_{11}^*, \tilde{x}_{2\ n-1}^*, ..., \tilde{x}_{n1}^*)$ to PP* can be calculated through the following iterative procedure where $i, j, k, l, s \in \mathbb{N}^+$, $j = 1, \ldots, n - i + 1$:

$$\tilde{x}_{1n}^* = \min_{k=1,\ldots,n} d_k$$

$\tilde{x}_{ij}^{*} = \min_{k=i,\ldots,i+j-1} (d_k - \sum \tilde{x}_{sl}^{*})$ where the last sum is over

$(s,l) \in \{\{s < i, l = 1,\ldots,n-s+1\} \cup \{s = i, l = j+1,\ldots,n-i+1\}\} \cap \{s+l \geq k+1\}$.

Exemplary Application of the Algorithm

In the following, we will illustrate the algorithm determining the optimal $\tilde{x}_{ij}^{*}$ for the example described in the introduction. In the example, we assumed n=3 and $d_k = (1,5,3)$.

We start the algorithm by setting $\tilde{x}_{13}^{*} = \min_{k=1,\ldots,3} d_k = 1$. The other $\tilde{x}_{ij}^{*}$ of the optimal solution $\tilde{x}^{*} = (\tilde{x}_{1n}^{*},\ldots,\tilde{x}_{11}^{*},\tilde{x}_{2\,n-1}^{*},\ldots,\tilde{x}_{n1}^{*})$ should optimally be determined backwards starting with $\tilde{x}_{1\,n-1}^{*}$.

$$\begin{aligned}
\tilde{x}_{12}^{*} &= \min_{k=1,2} \left\{ d_k - \sum \tilde{x}_{sl}^{*} \right\} \text{where the sum is over } (s,l) \in \{\{\ \} \cup \{(1,3)\}\} \cap \{s+l \geq k+1\} \\
&= \min\left\{ d_1 - \left(\tilde{x}_{13}^{*}\right), d_2 - \left(\tilde{x}_{13}^{*}\right) \right\} \\
&= \min\{d_1 - 1, d_2 - 1\} \\
&= \min\{0,4\} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\tilde{x}_{11}^{*} &= \min_{k=1} \left\{ d_k - \sum \tilde{x}_{sl}^{*} \right\} \text{where the sum is over } (s,l) \in \{\{\ \} \cup \{(1,2),(1,3)\}\} \cap \{s+l \geq k+1\} \\
&= \min\left\{ d_1 - \left(\tilde{x}_{13}^{*} + \tilde{x}_{12}^{*}\right) \right\} \\
&= 1 - 1 = 0
\end{aligned}$$

$$\begin{aligned}
\tilde{x}_{22}^{*} &= \min_{k=2,3} \left\{ d_k - \sum \tilde{x}_{sl}^{*} \right\} \text{where the sum is over } (s,l) \in \{\{(1,1),\ldots,(1,3)\} \cup \{\ \}\} \cap \{s+l \geq k+1\} \\
&= \min\left\{ d_2 - \left(\tilde{x}_{13}^{*} + \tilde{x}_{12}^{*}\right), d_3 - \left(\tilde{x}_{13}^{*}\right) \right\} \\
&= \min\{d_2 - 1, d_3 - 1\} \\
&= \min\{4,2\} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
\tilde{x}_{21}^{*} &= \min_{k=2} \left\{ d_k - \sum \tilde{x}_{sl}^{*} \right\} \text{where the sum is over } (s,l) \in \{\{(1,1),\ldots,(1,3)\} \cup \{(2,2)\}\} \cap \{s+l \geq k+1\} \\
&= \min\left\{ d_2 - \left(\tilde{x}_{13}^{*} + \tilde{x}_{12}^{*} + \tilde{x}_{22}^{*}\right) \right\} \\
&= \min\{d_2 - 1 - 0 - 2\} \\
&= \min\{2\} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
\tilde{x}_{31}^{*} &= \min_{k=3} \left\{ d_k - \sum \tilde{x}_{sl}^{*} \right\} \text{where the sum is over } (s,l) \in \{\{(1,1),\ldots,(1,3),(2,2),(2,1)\} \cup \{\ \}\} \cap \{s+l \geq k+1\} \\
&= \min\left\{ d_3 - \left(\tilde{x}_{13}^{*} + \tilde{x}_{12}^{*} + \tilde{x}_{22}^{*}\right) \right\} \\
&= \min\{d_3 - 1 - 0 - 2\} \\
&= 0
\end{aligned}$$

IV. CONCLUSION

The algorithm developed in this paper determines the cost minimizing optimal solution to a simplified unit commitment problem. The algorithm answers the questions which technology should be started up in a given period and how long it should produce. The algorithm can be easily used. In particular, optimization software is not necessary to determine the optimal solution. The algorithm also implies that dominated technologies should never be used for production.

Future work should generalize the algorithm. The inclusion of capacity limitations, partial load operation, and hydro storage and pump storage capacity would further increase the algorithm's value.

V. APPENDIX

The following lemma is applied to transform (DP) into (DP*):

Lemma:

Let $\Lambda = \left\{ y \in \mathbb{R}^n \,\middle|\, \exists x \in \mathbb{R}^n : y \le x, 0 \le Ax \le 1_n \right\}$ and $\Lambda^\pi = \left\{ y \in \mathbb{R}^n \,\middle|\, A_j y \le 1_{n-j+1},\ j \in N \right\}$

where A_j is defined as above. Then it holds: $\Lambda = \Lambda^\pi$.

Proof: By definition $\Lambda = \left\{ y \in \mathbb{R}^n \,\middle|\, \exists x \in \mathbb{R}^n : y_i \le x_i, i \in N \text{ and } 0 \le \sum_{\nu=j}^{n} x_\nu \le 1, j \in N \right\}$ and

$\Lambda^\pi = \left\{ y \in \mathbb{R}^n \,\middle|\, \sum_{\nu=\mu}^{\mu+j-1} y_\nu \le 1,\ j \in N \text{ and } \mu = 1,\dots,n-j+1 \right\}$.

Let $y \in \Lambda$, i.e. $\exists x \in \mathbb{R}^n : y_i \le x_i, i \in N$ and $0 \le \sum_{\nu=j}^{n} x_\nu \le 1, j \in N$.

It follows that $\sum_{\nu=\mu}^{\mu+j-1} y_\nu \le \sum_{\nu=\mu}^{\mu+j-1} x_\nu \le 1$ for $\mu = 1,\dots,n-j+1,\ j \in N$. The latter inequality follows from $\sum_{\nu=\mu}^{\mu+j-1} x_\nu = \sum_{\nu=\mu}^{n} x_\nu - \sum_{\nu=\mu+j}^{n} x_\nu \le 1$ for all $\mu = 1,\dots,n-j+1$ (using that $\sum_{\nu=\mu}^{n} x_\nu \le 1$, $\sum_{\nu=\mu+j}^{n} x_\nu \ge 0$ for all $\mu = 1,\dots,n-j+1$). Therefore, $y \in \Lambda^\pi$ and hence $\Lambda \subseteq \Lambda^\pi$.

Now, let $y \in \Lambda^\pi$. Then, it is to show that $\exists x \in \mathbb{R}^n : y_i \le x_i, i = 1,\dots,n$ and $0 \le \sum_{\nu=j}^{n} x_\nu \le 1, j \in N$.

We construct x applying the following algorithm:

Step 0: Set $x = y$. If $0 \le A \cdot x$, then "end". Otherwise, go to next step.

Step 1: Be $k^*(x)$ the largest index with property $\sum_{\nu=k^*}^{n} x_\nu < 0$.

Set $\begin{array}{ll} x'_\nu = x_\nu & \nu \ne k^* \\ x'_\nu = -\sum_{l=k^*+1}^{n} x_l & \nu = k^* \end{array}$

Step 2: Set $x = x'$. If $0 \le A \cdot x$ holds, then "end". Otherwise move back to step 1.

x' has the following properties:

(a) $x' \ge x$

(b) $k^*(x') < k^*(x)$ or $0 \le Ax'$

(c) $x' \in \Lambda^\pi$

The algorithm is finite because of (b). It follows from (a), (b), and (c) that the algorithm determines x with the desired properties. However, it remains to be shown that the three properties hold for x':

Ad (a): Follows from $x'_\nu = x_\nu$ for $\nu \ne k^*$ and $x'_{k^*} = x_{k^*} - \sum_{\nu=k^*}^{n} x_\nu \ge x_{k^*}$ (because $\sum_{\nu=k^*}^{n} x_\nu < 0$!).

Ad (b): If no $k^*(x')$ exists, then $0 \le A \cdot x'$. Otherwise, $k^*(x') < k^*(x)$ follows from $\sum_{\nu=k^*}^{n} x'_\nu = x'_{k^*} + \sum_{\nu=k^*+1}^{n} x'_\nu = -\sum_{\nu=k^*+1}^{n} x'_\nu + \sum_{\nu=k^*+1}^{n} x'_\nu = 0 \ge 0$.

Ad (c): We have to show that for $j \in N$ and $\mu = 1,...,n-j+1$ it follows $\sum_{\nu=\mu}^{\mu+j-1} x'_\nu \le 1$.

If $k^* < \mu$ or $k^* > \mu + j - 1$ this follows from $x'_\nu = x_\nu$ for $\nu \ne k^*$ and $x \in \Lambda^\pi$.

If $\mu \le k^* \le \mu + j - 1$ we have :

$$\sum_{\nu=\mu}^{\mu+j-1} x'_\nu = \sum_{\nu=\mu}^{k^*-1} x'_\nu + x'_{k^*} + \sum_{\nu=k^*+1}^{\mu+j-1} x'_\nu = \sum_{\nu=\mu}^{k^*-1} x_\nu - \sum_{\nu=k^*+1}^{n} x_\nu + \sum_{\nu=k^*+1}^{\mu+j-1} x_\nu = \sum_{\nu=\mu}^{k^*-1} x_\nu - \sum_{\nu=\mu+j}^{n} x_\nu$$

$\le \sum_{\nu=\mu}^{k^*-1} x_\nu$, because $\sum_{\nu=\mu+j}^{n} x_\nu \ge 0$ (Note: $\mu + j \ge k^* + 1 > k^*$!)

≤ 1, because $x \in \Lambda^\pi$.

Hence, $y \in \Lambda$ and $\Lambda^\pi \subseteq \Lambda$ which shows the lemma. □

Proof of Theorem 1:

(DP) can be written as follows:

$$\min < d, \lambda_0 >$$

$$\frac{-\lambda_0 - vc_i \cdot 1_n}{sc_i} \leq \frac{\lambda_i}{sc_i} \text{ and } 0 \leq A\frac{\lambda_i}{sc_i} \leq 1_n \text{ for } i \in S .$$

Applying the lemma with $y = \frac{-\lambda_0 - vc_i \cdot 1_n}{sc_i}$ and $x = \frac{\lambda_i}{sc_i}$ leads to the following equivalent representation:

$$\min < d, \lambda_0 >$$

$$A_j \frac{-\lambda_0 - vc_i \cdot 1_n}{sc_i} \leq 1_{n-j+1},\ i \in S,\ j \in N$$

Slightly rephrasing that expression and using that $A_j 1_n = j \cdot 1_{n-j+1}$ yields:

$$\min < d, \lambda_0 >$$

$$A_j(-\lambda_0) \leq (sc_i + j \cdot vc_i) \cdot 1_{n-j+1}, i \in S,\ j \in N$$

Theorem 1 follows from the definition of γ_j and the transformation $\tilde{\lambda}_0 = -\lambda_0$. □

Proof of Theorem 2:

We stated in theorem 2 that an optimal solution x^* to PP* can be calculated with the following algorithm:

$$\tilde{x}^*_{1n} = \min_{k=1,\ldots,n} d_k$$

$\tilde{x}^*_{ij} = \min_{k=i,\ldots,i+j-1} (d_k - \sum \tilde{x}^*_{sl})$ where the last sum is over

$$(s,l) \in \{\{s < i, l = 1,\ldots,n-s+1\} \cup \{s = i, l = j+1,\ldots,n-i+1\}\} \cap \{s+l \geq k+1\}.$$

Proof:

Suppose $\tilde{x}'^{1n} = (\tilde{x}'^{1n}_{1n},\ldots,\tilde{x}'^{1n}_{11}, \tilde{x}'^{1n}_{2\,n-1},\ldots,\tilde{x}'^{1n}_{n1})$ is an optimal solution with $\tilde{x}'^{1n}_{1n} < \min_{k=1,\ldots,n} d_k$. Note, that $\tilde{x}'^{1n}_{1n} > \min_{k=1,\ldots,n} d_k$ is impossible without the permissibility of partial load. In that case, it holds that

$$\sum_{j=1}^{n-1} \sum_{i=1}^{n-j+1} \tilde{x}'^{1n}_{ij} \cdot a_{ij} = d - \tilde{x}'^{1n}_{1n} \cdot a_{1n} > 0 .$$

As a consequence, we can construct indices meeting the following properties:

1. $\tilde{x}'^{1n}_{i_l j_l} > 0, l = 1,...,L$. This first property obviously ensures that all $\tilde{x}'^{1n}_{i_l j_l}$ are strictly positive.

2. $i_1 = 1, i_L + j_L - 1 = n$. This property ensures that the production schedule starts at the beginning and ends in the last period.

3. $i_l < i_{l+1}, l = 1,...,L-1$. The third property means that the schedule moves strictly to the right, each starting point is later than the one before.

4. $i_l + j_l < i_{l+2}$. This means that there must be a positive distance between the end of slice l and the beginning of slice $l+2$.

5. $i_l + j_l - 1 < i_{l+1} + j_{l+1} - 1, l = 1,...,L-1$. This property states that the end points also move to the right.

These indices are constructed as follows: firstly, define $j(i) = \max\left\{k \middle| k \in N, \tilde{x}'^{1n}_{ik} > 0\right\}$. Secondly, the first index is given by $i_1 = 1, j_1 = j(i_1)$. Following indices are determined by the minimum: $i_{l+1} = \min\left\{s \middle| s \in N, i_l < s, s + j(s) > i_l + j_l - 1\right\}$, $j_{l+1} = j(i_l)$. Finally, the procedure finishes with indices i_L and $j_L = j(i_L)$ if $i_L + j_L - 1 = n$.

Once these indices fulfilling (1) to (5) are constructed, it follows as an implication that $\sum_{l=1}^{L} a_{i_l j_l} = a_{1n} + \sum_{l=1}^{L-1} a_{i_{l+1}\, i_l + j_l - i_{l+1}}$.

$\gamma(t) = \min_{i=1,...,s}\left\{sc_i + t \cdot vc_i\right\}$ is both a monotone and concave function with respect to t. Using standard properties of such a function (Avriel 1976) we can show that:

$\sum_{l=1}^{L} \gamma_{j_l} \le \gamma_n + \sum_{l=1}^{L-1} \gamma_{i_l + j_l - i_{l+1}}$.

Hence, we can determine a new valid optimal solution:

$$\tilde{x}'^{1n\,1}_{i_l j_l} = \tilde{x}'^{1n}_{i_l j_l} - \min_{l=1,...,L} \tilde{x}'^{1n}_{i_l j_l}, l = 1,...,L$$

$$\tilde{x}'^{1n1}_{i_{l+1}\, i_l + j_l - i_{l+1}} = \tilde{x}'^{1n}_{i_{l+1}\, i_l + j_l - i_{l+1}} + \min_{l=1,...,L} \tilde{x}'^{1n}_{i_l j_l}, l = 1,...,L-1$$

$$\tilde{x}'^{1n\,1}_{1n} = \tilde{x}'^{1n}_{1n} + \min_{l=1,...,L} \tilde{x}'^{1n}_{i_l j_l}, l = 1,...,L$$

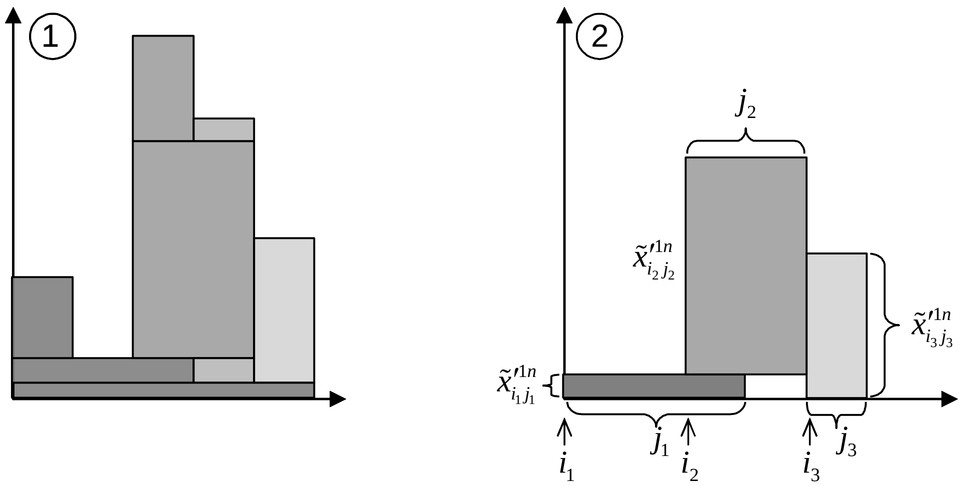
This procedure can be repeated until $\tilde{x}'^{1n m_{1n}}_{i_l j_l} = \min_{k=1,...,n} d_k$ for an m_{1n}. The procedure is finite because (i_{l^*}, j_{l^*}), defined by $\tilde{x}'^{1n}_{i_{l^*} j_{l^*}} = \min_{l=1,...,L} \tilde{x}'^{1n}_{i_l j_l}$, is not needed and remains zero in all following iteration steps. Because the number of possible index combinations is finite, the procedure ends after a finite number of steps.

The reasoning for the following values is analogous, based on the fact that all relevant components of vector $d - \sum_{(s,l)\ iterated} \tilde{x}^{*}_{sl} a_{sl}$ are strictly greater zero. □

An Example Illustrating the Proof of the Algorithm in Theorem 2

Figure 2 shows indices meeting the criteria 1. to 5. specified in the proof to the theorem. In the left figure (1), we have an exemplary problem with $n = 5$ periods and demand $d = (3,1,9,7,4)^T$. Furthermore, (1) has an exemplary production schedule which is sub-optimal as $\tilde{x}'^{1n}_{1n} < \min_{k=1,\ldots,n} d_k$. We start the production schedule for all capacity started in period 1 ($\tilde{x}'^{1n}_{1j}$) in dark grey and move step by step to a very light grey for all capacity started in the last period ($\tilde{x}'^{1n}_{51}$).

Figure 2: Indices Meeting Criteria 1. to 5.



Out of this schedule, we select three $\tilde{x}'^{1n}_{i_l j_l}$ satisfying 1. to 5. These are $\tilde{x}'^{1n}_{13} = \tilde{x}'^{1n}_{i_1 j_1}$, $\tilde{x}'^{1n}_{32} = \tilde{x}'^{1n}_{i_2 j_2}$, $\tilde{x}'^{1n}_{51} = \tilde{x}'^{1n}_{i_3 j_3}$. In (2), these are moved vertically on the X-axis and relabeled $\tilde{x}'^{1n}_{i_l j_l}$ with $l = 1,\ldots,3$. Do these selected $\tilde{x}'^{1n}_{i_l j_l}$ satisfy the proposed conditions 1. to 5.? They satisfy the first property as $\tilde{x}'^{1n}_{i_l j_l} > 0$ for all $l = 1,\ldots,3$. They also satisfy 2. as the first selected x starts in period one $i_1 = 1$ and the schedule ends in n (the last x is $\tilde{x}'^{1n}_{51}$, where $i_L + j_L - 1 = 5 + 1 - 1 = 5 = n$). The third property $i_l < i_{l+1}, l = 1,\ldots, L-1$ is also satisfied as $i_1 = 1 < i_2 = 3 < i_3 = 5$. In can be seen in the figure that the fourth property is also satisfied. It is also clear from the analytics as $i_1 + j_1 = 4 < i_3 = 5$. The fifth property is satisfied: $i_1 + j_1 - 1 = 3 < i_2 + j_2 - 1 = 4 < i_3 + j_3 - 1 = 5$.

As a result: $\sum_{l=1}^{L} a_{i_l j_l} = a_{1n} + \sum_{l=1}^{L-1} a_{i_{l+1}\, i_l + j_l - i_{l+1}}$

$$\sum_{l=1}^{L} a_{i_l j_l} = a_{13} + a_{32} + a_{51} = \begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\2\\1\\1\end{pmatrix}$$

$$= a_{1n} + \sum_{l=1}^{L-1} a_{i_{l+1}\; i_l + j_l - i_{l+1}} = a_{15} + a_{i_2\; i_1 + j_1 - i_2} + a_{i_3\; i_2 + j_2 - i_3} = a_{15} + a_{31} + a_{50}$$

$$= \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} + a_{50}$$

$$\sum_{l=1}^{L} \gamma_{j_l} = \gamma_3 + \gamma_2 + \gamma_1 \geq \gamma_n + \sum_{l=1}^{L-1} \gamma_{i_l + j_l - i_{l+1}} = \gamma_5 + \gamma_1 + \gamma_0$$

In the result, a generation slice of $\tilde{x}''^{1n}_{1n} = \min_l \tilde{x}'^{1n}_{i_l j_l}$ is added to $\tilde{x}'^{1n}_{1n}$ to get closer to the optimal schedule. $\tilde{x}''^{1n}_{13} = 0$ in the new solution. Furthermore, we know from the construction of the algorithm that $\tilde{x}''^{1n}_{13}$ will remain equal to zero for all future iterations.

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