Use of Zarka's Method at FHL

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Theory
Implementation in ANSYS
Experience

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Background, History

in nuclear industry:
elastic-plastic response required for life assessment under cyclic loading:

→ strain range $\Delta \varepsilon$
→ cyclic accumulated (ratchet-) strain $\varepsilon_{\text{acc}}$
→ displacements at end of life

20 years ago:
only rough estimates available as „simplified“ elastic-plastic methods (knock-down factors), without capturing effects of:

→ individual geometry of structure
→ kind of loading (thermal, displacement-, force-controlled, ...)
→ hardening of material
by end of seventies Zarka's method came up, claiming it was able to account for these effects

publication of some examples obtained by Zarka with Zarka's method revealed:

→ superior quality of results (approximations almost exact)
→ obtained with little numerical effort (few lin. elastic analyses)

however: method remained largely obscure:

→ description of theory appeared to be not complete
→ some assumptions seemed to be heuristically motivated
→ some applications outside Zarka's team showed poor results

question: why that ambiguous?

when advantageous?
Outline of Zarka's method

material model:
- yield surface must exist
- hardening required:
  » purely kinematic
  » linear or multilinear
- temperature dependence:
  only size of yield surface, i.e. \( \sigma_y \) (not \( E, \nu, E_t \))

⇒ "infinite" ratcheting ruled out:
  limit state = ES or PS,
  associated with "finite" ratcheting

simplified method:

only partial information:
- no evolution of \( \sigma, \varepsilon \) with cycles
- only post-shakedown quantities
- loss in accuracy (\( V_p, \sigma, \varepsilon \))

only reduced effort:
- direct method
- few elastic analyses sufficient
  + some "local" calculations
Specialisation, Modification

Zarka-method → specialisation, modifications → simplified theory of plastic zones (VFZT)

- moderate levels of strain (e.g. no deep drawing)
- isotropic material
- no strength differential effects (tensile = compressive)
- linear and bilinear kinematic hardening (bilin., trilin. $\sigma$–$\varepsilon$)
- Mises yield surface
- temperature-dependent yield stress
- mainly cyclic loading between 2 extreme states of loading
- logic of iterative improvement: identifying $V_p$, estimating TIV $Y$
Reformulation of “Exact” Plastic Theory

- uniaxial stress state
- bilinear \( \sigma - \varepsilon \)-diagram (kinem. hardening)
- monotonic loading

<table>
<thead>
<tr>
<th>kind of analysis</th>
<th>in plastic zone ( V_p )</th>
<th>in el. zone ( V_e ) (=( V - V_p ))</th>
<th>load applied</th>
</tr>
</thead>
<tbody>
<tr>
<td>exact (el-pl)</td>
<td>( \varepsilon_{el-pl} = \sigma_{el-pl}/E + \xi/C )</td>
<td>( \varepsilon_{el-pl} = \sigma_{el-pl}/E )</td>
<td>yes</td>
</tr>
<tr>
<td>fictitious elastic (f.el)</td>
<td>( \varepsilon_{f.el} = \sigma_{f.el}/E )</td>
<td>( \varepsilon_{f.el} = \sigma_{f.el}/E )</td>
<td>yes</td>
</tr>
<tr>
<td>difference</td>
<td>( \varepsilon_{el-pl} - \varepsilon_{f.el} = (\sigma_{el-pl} - \sigma_{f.el})/E + \xi/C ) ( \varepsilon^* = \rho + \xi/C )</td>
<td>( \varepsilon_{el-pl} - \varepsilon_{f.el} = (\sigma_{el-pl} - \sigma_{f.el})/E ) ( \varepsilon^* = \rho )</td>
<td>no</td>
</tr>
<tr>
<td>modif. elast. (mea)</td>
<td>( \Rightarrow \varepsilon^* = \rho \cdot (1/E^<em>) + \varepsilon_0 ) ( \text{with } 1/E^</em> = 1/E + 1/C ) and ( \varepsilon_0 = Y/C )</td>
<td>( \Rightarrow \varepsilon^* = \rho \cdot (1/E) )</td>
<td>initial strain ( \varepsilon_0 ) in ( V_p )</td>
</tr>
</tbody>
</table>

\( \Rightarrow \) residual state can be obtained by 1 mea with modif. elastic material data \( (E^*) \) and modif. loading \( (\varepsilon_0) \), if \( V_p \) and \( Y \) are known

\( \Rightarrow \) idea of Zarka-Method: estimation of \( V_p \) and \( Y \)
**Mon. Loading: Estimation of Y**

Here: bilinear \( \sigma - \varepsilon \)-diagram (lin. kinem. hardening)

\[ \sigma' - \text{space:} \]

- \( \xi = C \cdot \varepsilon_{\text{pl}} \)
- \( \xi \) is unknown; estimation not possible, since center \( \xi (\text{i.e. } \varepsilon_{\text{pl}}) \) also unknown

\[ \text{Mises yield condition:} \]

\[ f(\sigma', \xi, \sigma_y) \]

\[ f(Y, \sigma'_{\text{f.el}}, \sigma_y) \]

\[ Y \overset{\text{def}}{=} \xi - (\sigma' - \sigma'_{\text{f.el}}) \]

\[ \rho' \]

\[ \text{Y – space:} \]

- \( Y \) is unknown; but estimation possible, since center \( \sigma'_{\text{f.el}} \) is known

\[ \text{uniaxial stress:} \]

- \( Y \) exactly known by projection; only \( V_p \) unknown

\[ \text{multiaxial stress:} \]

- \( V_p \) and \( Y \) unknown; local estimation of \( Y \) sufficient, since field effects of \( Y \) are relieved compared to \( \sigma \) (=heuristic)
Mon. Loading: Algorithm

- fictitious elastic analysis (f.el) of max. loading
  - estimate extension of plastic zone \( V_p \)
  - estimate TIV \( Y \)
  - modify loading: real loading \( \rightarrow \) initial strain \( \varepsilon_0 \)
  - modify elastic parameters: \( E, \nu \rightarrow E^*, \nu^* \)
  - modified elastic analysis (mea)
    - elastic-plastic solution by superposition, e.g. \( \sigma_{el-pl} = \sigma_{f.el} + \rho \)
    - iterative improvement of \( V_p \) and \( Y \)

- numerical effort required:
  - few linear analyses (fictitious elastic and modified elastic)
  - local calculations
Iterative improvement of $V_p$:

$\sigma_{v}^{el-pl} > \sigma_y \leq \sigma_{ij}^{el-pl}$

with $\sigma_{ij}^{el-pl}$ from previous iteration

Iterative improvement of $Y$:

$Y - space$: $\sigma_{f-el}$

Projection $Y^* = -\rho'_i$

with $\rho_{ij}$ from previous iteration
Example 1: Mon. Loading: 2 Bars in Series

- hand calculation ⇒ analytical solution
- e.g. for specific configuration \((A_1/A_2, l_1/l_2; E_t/E)\):

\[ \frac{\varepsilon_{el-pl}}{\varepsilon_{f.el}} \]

\[ \frac{\sigma_{f.el}}{\sigma_y} \]

\( \Rightarrow \) iteration of \(V_p\) partly necessary
\( \Rightarrow \) exact results obtained after maximum 1 iteration
Example 2a: Mon. Loading: Beam of ideal I-Profile

- hand calculation ⇒ analytical solution; for specific $E_t/E$:

\[ \Delta w_0 \]

\[ \sigma \]
\[ \sigma_y \]
\[ E_t \]
\[ \varepsilon \]

\[ \frac{\varepsilon_{el-pl}}{\varepsilon_{f.el}} \]

1 iteration

exact

\[ \frac{\sigma_{f.el}}{\sigma_y} \]

\[ 1 \ 1.2 \ 1.4 \ 1.6 \ 1.8 \ 2.0 \ 2.2 \ 2.4 \]

\[ 1 \ 1.5 \ 2 \ 2.5 \ 3 \ 3.5 \]

→ after 1 iteration close to exact results
Example 2b: Mon. Loading: Beam of rect. Cross Section

Hand calculation possible for $E_t=0$ (via initial stress):

\[ \Delta w_0 \]

\[ \sigma \]

\[ \sigma_y \]

\[ E \]

\[ E_t=0 \]

\[ \varepsilon \]

\[ \varepsilon_{el-pl} \]

\[ \varepsilon_{f.el} \]

\[ \frac{\varepsilon_{el-pl}}{\varepsilon_{f.el}} \]

\[ \frac{\sigma_{f.el}}{\sigma_y} \]

\[ 1 \text{ iteration} \]

\[ 1.5 \]

\[ 2 \]

\[ 2.5 \]

\[ 3 \]

\[ 3.5 \]

\[ \text{exact} \]

\[ \text{without iteration} \]

\[ \text{mid length} \]

\[ \text{top surface} \]

\[ \text{exact} \]

\[ \text{without iteration} \]

\[ \text{clamped} \]

\[ \text{middle surface} \]

\[ \text{plastic zone} \]
Implementation in FE-codes

Implementation in any commercial FE code (ANSYS, ABAQUS, ADINA, ...) usually not easy:
- modification of elastic material properties and of loading (initial strain) required during iterative solution
- access to elastic material properties and application of initial strain usually inconsistent: $E \rightarrow$ element, $\varepsilon_0 \rightarrow$ Gauss or nodal point
- exact definition of $V_p$ somewhat arbitrary ($V_p$, if $\sigma_v > \sigma_y$ in $> 50\%$ of Gauss-points)

Implementation at FHL in ANSYS:
for general structures (1D, 2D, 3D):
- bilinear $\sigma - \varepsilon$
- $T$-dependent $\sigma_y$
- mon. + cyclic loading (2 states of loading)

Extension for specific structures (trusses, beams):
- trilinear $\sigma - \varepsilon$
- increm. projection in ES, if $> 2$ states of loading
## Ways of Implementation in ANSYS

1. **VEFZOT: APDL-Macro (=ANSYS parameter language; slow)**  
   initial strain simulated by:  
   * thermal strain at nodal points  
   * orthotropic coefficient of thermal expansion \( \rightarrow \) deviatoric  
   * orthotropic directions changing from el. to el. \( \rightarrow \) sometimes poor qual.

2. **Userpl.f: FORTRAN user subroutine (compiling and linking required)**  
   initial strain simulated by:  
   * inelastic strain at Gauss points  
   * formally equilibrium iterations performed, although not necessary

3. **IS-Macros (APDL-Macro) or Ustress.f (FORTRAN-subroutine):**  
   initial strain simulated by initial stress at Gauss points

4. **Particular implementation for trusses and bending beams:**  
   initial strain simulated by \( ts \) and \( \Delta t \) across section
Example 3: Mon. Loading: Hollowed Plate

displacement + Mises effective stress

axial strain

\[
\sigma = \varepsilon E + \varepsilon_y \sigma_y
\]

plastic zone

exact

VFZT

\[
\sigma_y
\]

exact VFZT

VFZT
Example 4: Mon. Loading: Crack in a Plate

- Monotonic loading

Mises effect. stress:
- Fictitious elastic
- VFZT, 3. mea
- Evolutive

Plastic zone
Cyclic Loading: ES: Estimation of $Y$

Load histogram:
- Constant load
- Cyclic load
- Time

Elastic shakedown (ES): if $\Omega$ exists at each point of structure

$V_p$, if $Y^*=-\rho'$ is outside $\Omega$:

$\sigma_{\text{f.el}} \min ^* \Omega$

$\sigma_{\text{f.el}} \max ^* \Omega$

$Y^*=-\rho'$
Example 5: ES: 2 Bars Parallel

The image shows a diagram of two bars in parallel, subjected to a load $F$. The time is plotted against the number of cycles.

**Numerical effort required to achieve ES:**
- **Exact:** 1000 cy. * 10 loadsteps * 5 iterations
  - ~ ca. 50 000 linear elast. anal.
- **VFZT:** 2 f.el. + 2 mea = 4 linear elast. anal.

**Identical results:**

The graph on the right shows the strain accumulation $\varepsilon_{el-pl}/\varepsilon_y$ against the number of cycles. The diagram illustrates elastic shakedown and strain accumulation.
Example 6: ES: Thermal Stratification

\[ \Delta T \]

\[ \sigma \]

\[ \varepsilon \]

\[ E_t \]

\[ \varepsilon_y \]

\[ \Delta T \]

\[ \varepsilon_p \]

\[ \varepsilon_y \]

VFZT:

good approxim.

(i.e. 3 linear elastic analyses)

fictitious elastic analysis

evolutive analysis

evolutive

shakedown

VFZT
Cyclic Loading: PS: Estimation of $\Delta Y$

**load histogram:**
- **const. load**
- **max**
- **min**
- **cyclic load**

**range values:** $V_p$, if $\Delta \sigma_v > 2\sigma_y$:

**plastic shakedown (PS):** if there is no intersection at least at 1 point of structure
Cyclic Loading: PS: Estimation of $Y_{\text{mean}}$

**mean values: $V_p$, if $\sigma_{v,\text{min}} > \sigma_y$ or $\sigma_{v,\text{max}} > \sigma_y$:**

**in $V_p, \Delta$:**

projection of $Y^* \pm \frac{1}{2} \Delta Y$ on max (or on min, if closer):

**in $V_e, \Delta$:**
Example 7: PS: Beam under $N$, $\Delta w_0$

- Plastic shake down
- $\sigma$ vs $\varepsilon$
- Evolutive analysis
- Fictitious elastic analysis
- VFZT

Graph shows evolution of strain $\varepsilon_{acc}$ and total strain $\Delta \varepsilon$ over time.
Example 7 (ctd.): PS: Beam: $\Delta \varepsilon$

- **fictitious elastic**
- **VFZT:**
  - 1 mea
  - 2 mea
  - 3 mea

- **plastic shake-down**
- **evolutive:** 15 cycles
Example 7 (ctd.): PS: Beam: $\varepsilon_{\text{acc}}$

- Fictitious elastic
- VFZT:
  1 mea
  2 mea
  3 mea
  4 mea
  5 mea
- Evolutive: 15 cycles
- Plastic shakedown
- Time
Example 8: PS: Cylindrical Shell under Axial Force and Temperature

\[ \sigma_{N_{\text{f.el}}}/\sigma_y = 0.8 \]
\[ \sigma_{T_{\text{f.el}}}/\sigma_y = 2.2 \] (\( \rightarrow \) PS)

\[ E_t/E = 0.03 \]
\[ \nu = 0.3 \]
Example 8 (ctd.): PS: Cylindrical Shell: axial $\Delta \varepsilon$

fictitious elastic

Zarka’s method

1 mea 2 mea 3 mea

evolutive (100 cycles)
Application: Ratcheting-Interaction Diagrams

which combination of load parameters yields ES, PS, R for specific configuration of structure and loading?

→ by shakedown theorems
→ no post-shakedown quantities

Zarka's method:

→ ES, PS known a priori
→ R not possible: subregions ES₂, PS₂
→ post-shakedown quantities (Δε, ε_{acc})
Example 9: Ratch-Inter: 2 Bars Parallel

Hand calculation for any configuration of loading (T, F) and hard. param. \((E_t/E)\):

\[
\frac{\sigma_p}{\sigma_y} = 0.2
\]

\[
\frac{\sigma_p}{\sigma_y} = 0.5
\]
Example 10: Ratch-Inter: 2 Bars Parallel (Trilinear)
Example 11: Ratch-Inter: Bree Problem

Hand calculation for any configuration of loading \((T, P)\) and hard. param. \((E_t/E)\):

\[\sigma_p/\sigma_y \approx 0.4\]

\[\varepsilon_{el-pl}/\varepsilon_y\]

\[\approx \text{exact after 2 iterations}\]

\[\rightarrow \text{close to exact after 2 iterations}\]
Example 12: Ratch-Inter: Beam under $N$, $\Delta w_0$

accumulated strain:

$\frac{\sigma_{f,el}}{\sigma_y}$

$\frac{\Delta w_0}{N}$

max. load level

time

max. Belastungszustand

$\frac{\varepsilon_{acc}}{\varepsilon_y}$:

- 10,0-12,0
- 8,0-10,0
- 6,0-8,0
- 4,0-6,0
- 2,0-4,0
- 0,0-2,0
1. Benchmark CUT-FHL: Thick-walled Cylinder

thick walled cylinder under internal pressure

mon. loading, load level $\sigma_{v}^{f.el}/\sigma_{y}=1.665$

material: lin. kin. hardening

<table>
<thead>
<tr>
<th>combination</th>
<th>$\nu$</th>
<th>$E_t/E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.49</td>
<td>0.001</td>
</tr>
<tr>
<td>B</td>
<td>0.49</td>
<td>0.1</td>
</tr>
<tr>
<td>C</td>
<td>0.3</td>
<td>0.001</td>
</tr>
<tr>
<td>D</td>
<td>0.3</td>
<td>0.1</td>
</tr>
</tbody>
</table>
1. Benchmark CUT-FHL (ctd.): Thick-walled Cylinder

results:

combinations A, B, D: all residual stress- and strain-components after 6 measurements close to exact values (difference <1%);

combination C (E_t/E=0.001, ν=0.3):

no improvement with more measurements → loss in accuracy: why?
2. Benchmark CUT-FHL: Hertz Contact

- Hertz contact (plane stress)
- Stationary cyc. loading (ES); load level $\sigma_v^{f,el}/\sigma_y = 1.388$
- Lin. kin. hardening $E_t/E = 0.1$
2. Benchmark CUT-FHL (ctd.): Hertz Contact

results:
residual stresses along vertical section:

\[ \rho_x \]  
\[ \rho_y \]  

\[ \text{evolutive} \]  

\[ \rightarrow \text{slow approximation to exact solution: why?} \]
3. Benchmark CUT-FHL: Bree Problem

- Thin-walled cylinder under internal pressure and radial temperature gradient
- Cyclic loading (ES, PS)
- Linear kinematic hardening: $E_t/E = 0.1$, T-dependent yield stress

<table>
<thead>
<tr>
<th>combination</th>
<th>Sa</th>
<th>Sb</th>
<th>Sc</th>
<th>Pb</th>
<th>Pc</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_p^{f,el}/\sigma_y$</td>
<td>0.7</td>
<td>0.2</td>
<td>0.8</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>$\sigma_t^{f,el}/\sigma_y$</td>
<td>1.2</td>
<td>1.9</td>
<td>1.9</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>
3. Benchmark CUT-FHL (ctd.): Bree Problem

\( \sigma_y = 200 \) at inner and outer surface for min. load level (no temperature),
\( \sigma_y = 161.1 \) at inner surface and 238.9 at outer surface for max. load level
(fully developed T-gradient)

results for combination Pc:
- range values = exact after 2 mea
- accum. state at minimum load level:

\[ \varepsilon_{\text{acc,hoop}} \]

\[ \varepsilon_{\text{acc,axial}} \]

\( \rightarrow \) \( \varepsilon_{\text{acc}} \) reasonable, but not superior: why?
Further Development: Non-radial Loading

3-Bar Problem:

Nozzle:

Moving Load:
Zarka's Method (and it's modifications by FHL) is a simplified method to provide post-shakedown quantities in ES and PS by direct means ($\Delta \varepsilon$ for fatigue, $\varepsilon_{acc}$ for ratcheting-assessment, displacements).

is implemented in commercial FE-program ANSYS for monotonic and cyclic loading.

examples show in case of radial loading:
- good approximation to exact solution
- at small numerical effort (few linear elastic analyses)
- in particular if directional redistribution is not strongly developed
- range-values (PS) usually better than mean values

benchmarks between CUT and FHL raised some questions.