Application of the Simplified Theory of Plastic Zones on APDL-level

Examples for basic understanding of the method using ANSYS Mechanical APDL

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Introduction

These examples were developed to offer an insight into the practical application of the simplified theory of plastic zones (STPZ). The content is based on the first four chapters of the book 'Simplified Theory of Plastic Zones' by H. Hübel [1], also published in German [2].

In fact, an ANSYS user-subroutine programmed in FORTRAN already exists allowing the application of the STPZ for complex structures. Various material properties like temperature-dependent material and multilinear kinematic hardening can be considered.

The examples used here to demonstrate the STPZ are very elementary. Thus, certain simplifications can be made reducing the algorithm to the principal functions. The examples consist of a maximum of two elements with homogenous stress condition and no shear stresses. Linear kinematic hardening with constant yield stress is assumed. The employment of unit lengths and consistency in the units used for material parameters and loading allows the discussion of results without using units.

A short explanation of the method and the equations used here is given before the examples are discussed, but study of the corresponding sections in [1] is strongly recommended. All assumptions made in [1] are adopted without further reference. For easier comparison, the numbering of equations in the book is given additionally on the right.

The reader is asked to use this manual as follows: After internalising the basic method behind the STPZ, different examples with an increasing level of difficulty are offered to be tested. For each example, the required ANSYS Parametric Design Language (APDL) code to analyse this problem with either the STPZ or an incremental analysis is given in the annex, allowing the reader to follow the input step by step.

Although knowledge of using ANSYS and especially APDL is required, a basic knowledge may be sufficient, as the reader is not asked to write any code unless modifications need to be made. However, at some point the reader might want to enter APDL commands to study the results, as they are not always included. The examples using the STPZ usually save all significant values as parameters, which can be listed.

The first chapter addresses examples underlying monotonic loading, which is in fact not the purpose of the STPZ but might facilitate the understanding. The second chapter is concerned with cyclic loading.

This work was developed as coursework for part of the author's studies in Civil Engineering at the BTU Cottbus-Senftenberg.

List of Symbols

Name	Symbol in document	Labelling in APDL code
Yield stress	f_y	`fy`
Young's modulus	E	`E`
El.-pl. hardening modulus	E_t	`Et`
Plastic hardening modulus	$C = \frac{E \cdot E_t}{E - E_t}$	`C`
Poisson's ratio	ν	`nu`
Strain	ε	
Stress	σ	`s, s_elpl`
Residual stress	ρ	`rho`
Residual strain	ε^*	
Equivalent stress (von Mises)	σ_v	`seqv`
Backstress	ξ	
Deviatoric stress	σ'	`sdev`
Fictitious elastic stress	σ^{fel}	`s_fel`
Transformed internal variable	Y	`Y`
Modified parameter E	E^*	`Emod`
Modified parameter ν	ν^*	`numod`
Initial stress	σ_0	`sig0`
Initial strain	ε_0	
Displacement	u	`u`
Range values	$\Delta \ldots$	D…
Stress at extreme load	$\sigma_{max/min}$	`s_max, s_min`
TIV at mean load	Y_m	`Y_m`
Angles in deviatoric space	$\alpha_{min/max}$ $\beta_{min/max}$	`alpha` `beta_min, beta_max`
Parameters	a, b	`a, b`

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1 Monotonic Loading

1.1 STPZ at monotonic loading

To visualise the elastic-plastic behaviour of material usually the deviatoric stress space is used. The basic concept of the STPZ is to transfer the content to the space of the transformed internal variable (TIV) Y, where Y is the difference between backstress ξ and deviatoric residual stress ρ'. (Fig. 1)

(1 - 1) $$Y_i = \xi_i - \rho_i'$$ Eq. 3.1

The advantage for estimating resulting stress and strain values lies in the fact that the centre of the Mises circle is initially known from only one fictitiously elastic calculated analysis. With the additional knowledge of the diameter of the Mises circle, the TIV can be estimated by projecting the negative deviatoric residual stress on the yield surface. For radial loading this projection will lead to the correct answer for the resulting elastic-plastic stresses if no directional stress redistribution occurs and the plastic zone, i.e. the elements in which plastic straining occurs, is correctly identified.

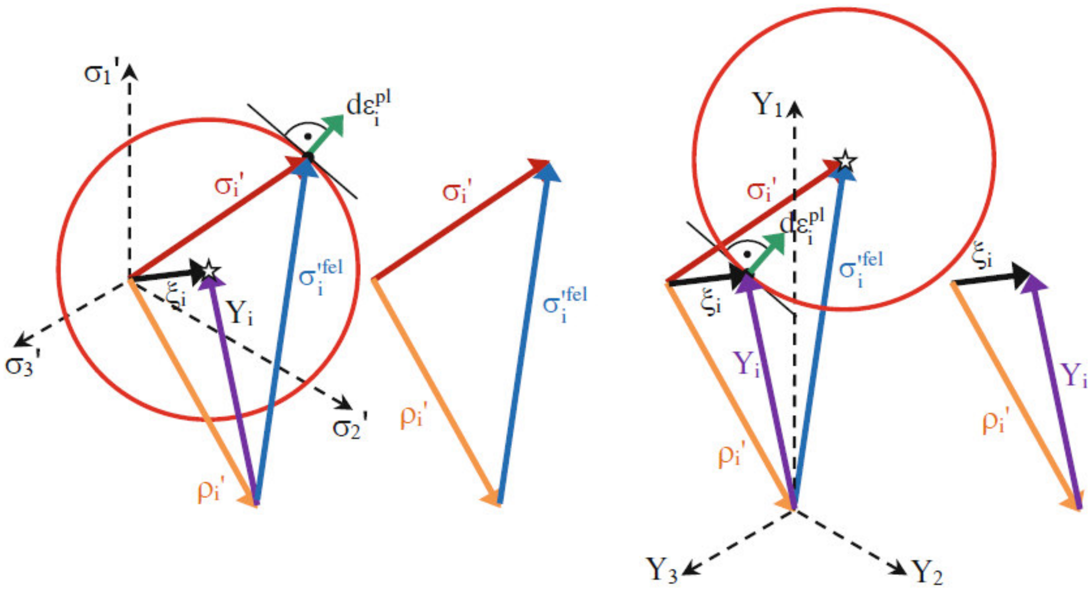


Fig. 1 Mises yield surface […] *left* in the space of the deviatoric stresses; *right* in the space of the TIV (Fig. 3.1 in [1])

For non-radial loading, the direction of Y is not known at the beginning. Therefore, an iteration loop is necessary to improve the result from step to step. Y can be estimated in step n by taking the deviatoric fictitious elastic stress and subtracting a vector of the size of the yield stress f_y with the direction of the deviatoric elastic-plastic stress of the previous step $(n-1)$.

(1 - 2) $$Y_i^{(n)} = \sigma_i'^{fel} - \sigma_i'^{(n-1)} \left(\frac{f_y}{\sigma_v^{(n-1)}} \right)$$ Eq. 3.108

As the elastic-plastic stress has not been calculated before the first iteration, it is initialised with the fictitious elastic stress.

(1 - 3) $$\sigma_i'^{(0)} = \sigma_i'^{fel}$$ Eq. 3.109

By reformulating the yield condition and the development of the plastic strain increment using only the fictitious elastic stress and the TIV, a new material law for the residual state can be established.

(1 - 4) $$\varepsilon_i^* = \left(E_{ij}^*\right)^{-1} \rho_j + \varepsilon_{i,0}$$ Eq. 3.12

In this $\varepsilon_{i,0}$ is the initial strain, which is applied to the structure after eliminating all other loads like forces, displacements and temperature loads. The initial strain is known from the TIV.

(1 - 5) $$\varepsilon_{i,0} = \frac{3}{2C} Y_i$$ Eq. 3.14

The modified elasticity matrix E_{ij}^* is obtained by replacing the Young's modulus E and the Poisson's ratio ν in Eq. 1.4 or rather Eq. 1.5 in [1] with modified material parameters E^* and ν^*.

(1 - 6) $$E^* = E_t$$ Eq. 3.15

(1 - 7) $$\nu^* = \frac{1}{2} - \frac{E_t}{E}\left(\frac{1}{2} - \nu\right)$$ Eq. 3.16

Alternatively, the inverted form of Eq. (1 - 4) can be given using initial stresses $\sigma_{i,0}$.

(1 - 8) $$\rho_i = E_{ij}^* \, \varepsilon_j^* + \sigma_{i,0}$$ Eq. 3.21

(1 - 9) $$\sigma_{i,0} = -\frac{3}{2C}\frac{E^*}{1+\nu^*} Y_i$$ Eq. 3.23

A modified elastic analysis (MEA) can then be performed to calculate the residual state of the structure using the modified material law and initial stresses or strains as given above for zones in which plastic straining occurs. Therefore prior to each MEA the structure is divided into the plastic zone V_p ($\sigma_v \geq f_y$) and the elastic zone V_e ($\sigma_v < f_y$). In the elastic zone no initial stresses nor strains are applied. The material parameters remain E and ν.

(1 - 10) $$V_e^{(n)} = \left\{\underline{x} \mid \sigma_v^{(n-1)} < f_y\right\}$$ Eq. 3.65

(1 - 11) $$V_p^{(n)} = \left\{\underline{x} \mid \sigma_v^{(n-1)} \geq f_y\right\}$$ Eq. 3.64

The result of the MEA is the residual stress state. Superposition of the fictitious elastic state and the residual state produces the elastic-plastic state (see Fig. 1). This is not only valid for stresses and strains but also for displacements, section forces and bearing forces.

(1 - 12) $$\sigma_i = \sigma_i^{fel} + \sigma_i^{MEA}$$ Eq. 3.24

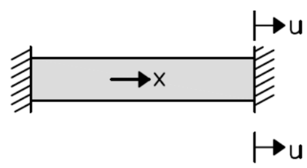
(1 - 13) $$\varepsilon_i^{el-pl} = \varepsilon_i^{fel} + \varepsilon_i^{MEA}$$ Eq. 3.25

Because the TIV is only estimated in Eq. (1 - 2) the exact result might only be approximated. An iterative improvement can be possible by repeating the procedure using the results of the previous step. In addition, the assignment to either elastic or plastic zone may be changed when comparing $\sigma_v^{(n-1)}$ to the yield stress.

1.2 Example M.1 Element with uniaxial stress state

The intention of this first, very simple example is the understanding of the STPZ algorithm. In fact, no decrease in computational effort is achieved here because an elastic-plastic analysis using linear kinematic hardening produces the correct result in one loadstep without substeps.

The structure consists of a single LINK element with one fixed end (node 1) and a displacement-controlled parallel load at the other end (node 2). The strain caused by u (Eq. (1 - 14)) is 1.5-times the elastic limit load because unit length is used.



(1 - 14) $$u = \frac{f_y}{E} \cdot 1.5$$

In preparation for the modified elastic analysis, the modified material parameters E^* (`Emod`) has to be calculated and is then defined as property of the modified material 2. The parameter ν^* (`numod`) is irrelevant for LINK elements.

The first step is to calculate the fictitious elastic stress state as it is needed in every modified elastic analysis and in the subsequent superposition. The equivalent stress σ_v^{fel} and the axial component of the deviatoric stress σ'^{fel} are saved as parameters `seqv_fel` and `sdev_fel`. Prior to the modified elastic analyses, the fictitious elastic state is saved as loadcase 1 for later use. The displacement-controlled force at node 2 has to be set to zero.

As the magnitude of the fictitious elastic stress ($\sigma_x^{fel} = 1.5 \cdot f_y$) is already known, plastic straining must occur in this element and the element can be assigned to the plastic zone.

According to chapter 3.1.4 in [1] modifications in calculating the TIV and the initial stress for uniaxial stress state must be made.

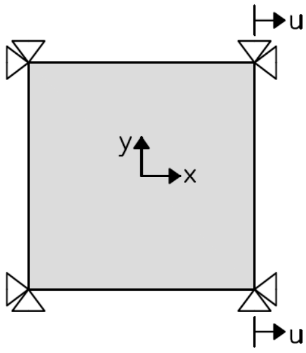
(1 - 15) $$Y = \frac{3}{2} Y_x$$ Eq. 3.37

(1 - 16) $$\sigma_0 = -\frac{1}{C} E^* Y$$ Eq. 3.41

The initial stress is applied by using the `inistate`-command and the material data of this element is changed to the modified material 2 using `mpchg`. The analysis produces the residual state (now stored in loadstep 1). Adding the fictitious elastic state results in the elastic-plastic state. The correct solution is obtained after the first iteration because the loading is radial and the plastic zone is known from the beginning.

1.3 Example M.2 Element with plane stress state

In this example, the algorithm is extended to a second stress component as the model underlies a plane stress condition ($\sigma_z = 0$). The structure of a single PLANE element is subjected to a displacement-controlled load in the x-direction. In the y-direction strains are prohibited ($\varepsilon_y = 0$), but a Poisson's ratio of $\nu \neq 0$ will lead to stress component $\sigma_y \neq 0$. This is inevitable when plastic behaviour occurs (see Sect. 3.6.1 in [1]).



The procedure remains the same as in example M.1 except for the calculation of the initial stresses for the plane stress condition (see Sect. 3.1.5 in [1]).

(1 - 17) $$\sigma_{x,0}^{2D} = \frac{\sigma_{x,0}^{3D} + \nu^* \sigma_{y,0}^{3D}}{1 - \nu^*}$$ Eq. 3.50

(1 - 18) $$\sigma_{y,0}^{2D} = \frac{\nu^* \sigma_{x,0}^{3D} + \sigma_{y,0}^{3D}}{1 - \nu^*}$$ Eq. 3.51

Using the same material parameters as before ($E/E_t = 0.05$, $\nu = 0$) one will see that the results after the first iteration differ up to 90 % from the exact solution. Therefore, an automatic loop for an arbitrary number of iterations is introduced to improve the results of the STPZ. Fig. 2 shows the development of elastic-plastic stress in x and y direction after each of the first 25 modified elastic analysis.

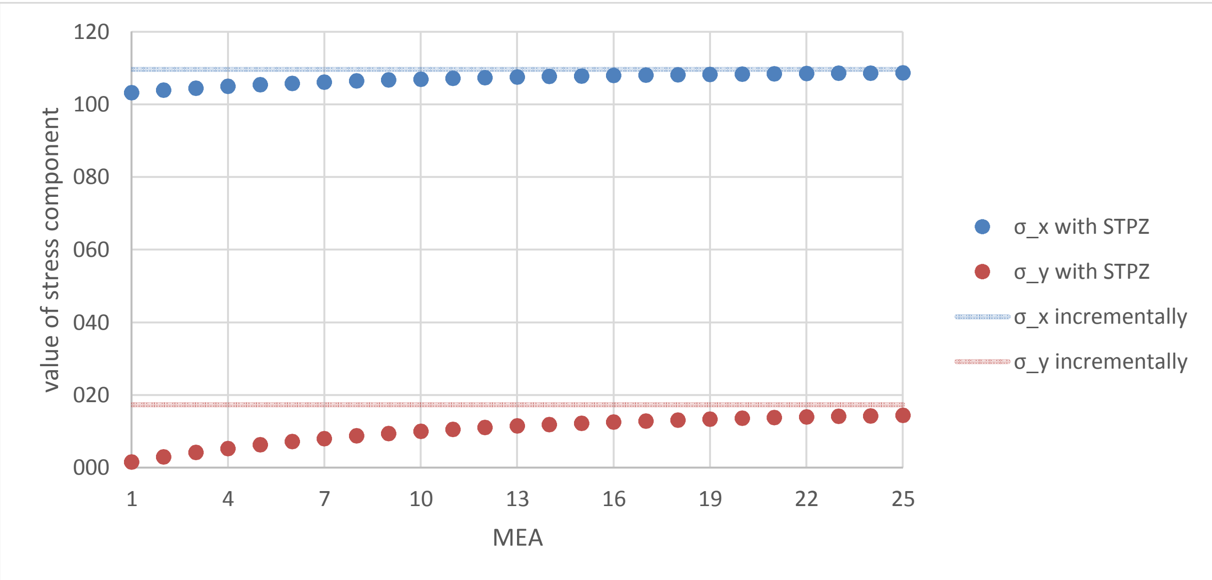
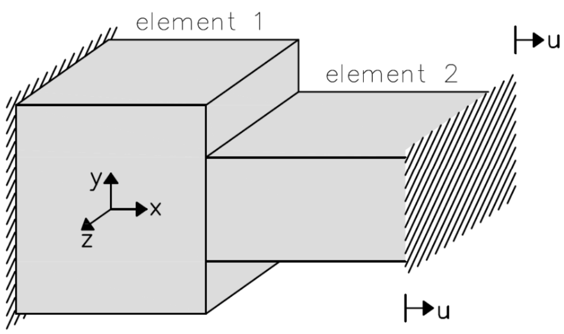


Fig. 2 Elastic-plastic stress results of the STPZ at each MEA from 1 to 25 for example M.2

In an incremental analysis the result depends on the number of substeps. The results of the STPZ even after the 25th iteration still differ slightly from the values obtained by an incremental analysis using 100 substeps. This is due to directional stress redistribution. When plastic straining occurs, the loading is no longer radial as the Poisson's ratio shifts towards $\nu^{pl} = 0.5$ because of plastic incompressibility. This weakness of the STPZ has to be considered, especially when small Poisson's ratios are used.
Comparing the results of the STPZ for this structure using $\nu^{el} = 0.499$ instead of 0, the differences from the exact values are less than 0.1 % after the first MEA.

1.4 Example M.3 Tension bar

In this last example for monotonic loading, the algorithm is extended to work for a multiple number of elements and for all three normal stress and strain components.
The structure consists of two neighbouring three-dimensional SOLID elements where the height of the right element is only half the height of the left one. All other dimensions (length and width) are the same. The degrees of freedom in the z-direction are coupled so the strain in this direction must be the same in both elements (generalised plane strain condition). A displacement-controlled load u is applied at the right end of the structure (see Sect. 3.6.2 in [1]).



After the fictitious elastic analysis is performed, the stresses are saved using a loop through all elements. The individual stress components are named using indices 1 to 3. This enables us to use a loop through all components for the estimation of the TIV and for the calculation of the initial stress components.
In both elements the fictitious elastic equivalent stress is greater than the yield stress. After the application of the initial stresses and changing to material 2 for both elements, the analysis is solved to get the residual state. This is then superimposed on the fictitious elastic state.
From the second MEA only element 2 remains plastic. For element 1 the initial stresses are deleted and the material parameters are reversed. The result is further improved in the following MEAs. Fig. 3 shows the results for the elastic-plastic stress in the right element after each of the first ten MEAs.

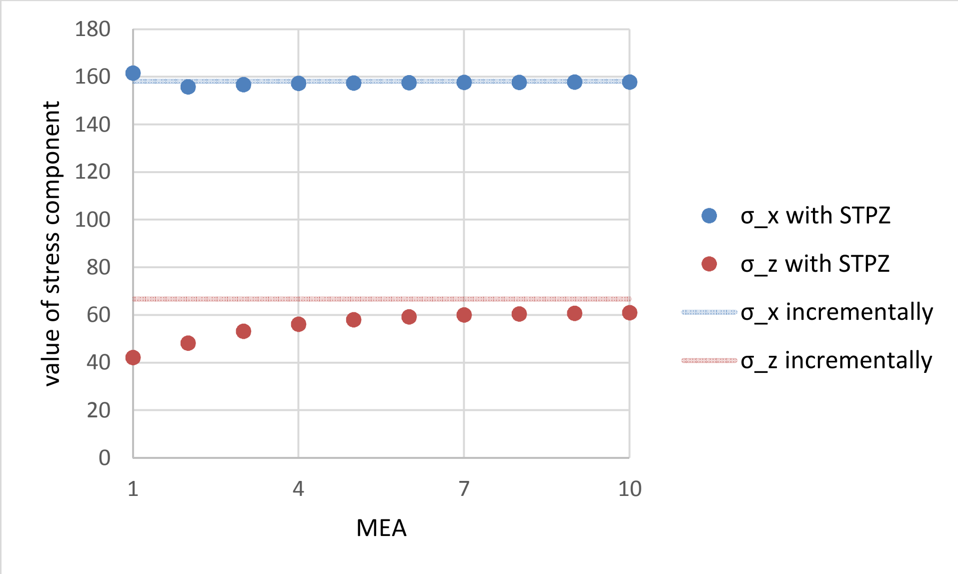


Fig. 3 Elastic-plastic stress results of the STPZ at each MEA from 1 to 25 for example M.3

The stress component in the x-direction differs less than 1% from the incremental result after the third MEA. The stress component in the z-direction still differs about 9% from the exact result even after the eighth MEA. The results for the strain components are similar.

2 Cyclic Loading

2.1 STPZ at cyclic loading

Unlike for monotonic loading, the resulting stresses and strains in structures undergoing cyclic loading often require high computational effort, because the result can often only be obtained after analysing a high number of cycles with many substeps using nonlinear material law. The advantage of the STPZ is that it enables calculation of the stress and strain range and the accumulated values using only elastic material law in a significantly reduced number of analyses.

Cyclic loading can be considered as an alternation of the load level between two loading conditions, called minimum and maximum load, regardless of magnitude and sign. The difference between two fictitious elastic stress states is the fictitious elastic stress range.

(2 - 1) $$\Delta\,\sigma_i^{fel} = \sigma_{i,max}^{fel} - \sigma_{i,min}^{fel}$$ Eq. 4.2

The nature of the state of shakedown can be determined by comparing the equivalent stress of the fictitious elastic stress range to the yield stress (see Sect. 4.1 in [1]). If twice the yield stress is not exceeded in any location of the structure ($\Delta\sigma_v^{fel} \leq 2f_y$) elastic shakedown (ES) will occur. For plastic shakedown (PS) the resulting strain range will have a plastic portion.

a) Plastic Shakedown

If at any point of the structure the fictitious elastic equivalent stress range exceeds twice the yield stress then plastic shakedown occurs.

(2 - 2) $$\Delta\sigma_v^{fel} > 2\,f_y \quad \exists \underline{x} \in V$$ Eq. 4.151

The elastic-plastic stress and strain ranges can be calculated analogous to monotonic loading using the corresponding range values (see Sect. 4.2 in [1]). The TIV is similarly estimated and improved in each iteration by projecting the negative deviatoric residual stress range at the Mises circle with a radius of $2f_y$ (see Fig. 4).

(2 - 3) $$\Delta Y_i^{(n)} = \Delta\sigma_i'^{fel} - \Delta\sigma_i'^{(n-1)}\left(\frac{2f_y}{\Delta\sigma_v^{(n-1)}}\right)$$ Eq. 4.17

The initial stress or strain ranges can be deduced from the previous chapter.

(2 - 4) $$\Delta\varepsilon_{i,0} = \frac{3}{2C}\Delta Y_i$$ Eq. 4.7

(2 - 5) $$\Delta\sigma_{i,0} = -\frac{3}{2C}\frac{E^*}{1+\nu^*}\Delta Y_i$$ Eq. 4.8

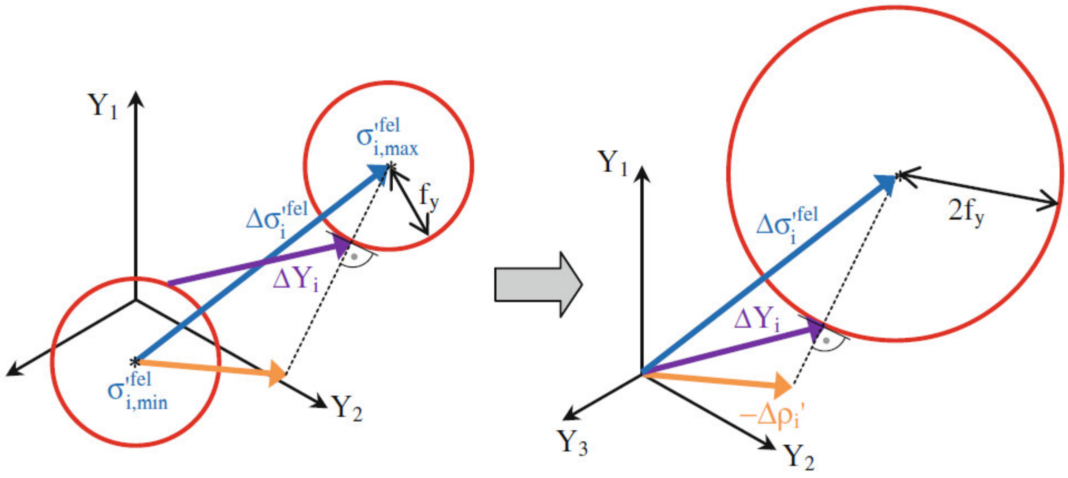


Fig. 4 Estimation of the TIV-range at cyclic loading in the space of the TIV (Fig. 4.2 in [1])

The residual stress and strain ranges are determined in an analysis using modified elastic material law and modified material parameters E^* and ν^* and by applying the initial load.

(2 - 6) $$\Delta\varepsilon_i^* = \left(E_{ij}^*\right)^{-1}\Delta\,\rho_j + \Delta\varepsilon_{i,0}$$ Eq. 4.5

(2 - 7) $$\Delta\rho_i = E_{ij}^*\Delta\,\varepsilon_j^* + \Delta\sigma_{i,0}$$ Eq. 4.6

The elastic-plastic range state is then received by superposition of the fictitious elastic and residual range states.

(2 - 8) $$\Delta\sigma_i = \Delta\sigma_i^{fel} + \Delta\rho_i$$ Eq. 4.14

To determine the accumulated strains in cyclically loaded structures, the estimation of the TIV depends on the nature of the state of shakedown and on the subvolume that the location (element) has been assigned to in the current iteration step (see Sect. 4.6 in [1]). The classification is made prior to each modified elastic analysis (n) so the values are taken from the previous step $(n-1)$. Before the first MEA, the results of the fictitious elastic analysis are regarded as the elastic-plastic results for step $n-1=0$.

For PS the elements of the structure can be assigned to one of the three following zones:

In the elastic zone V_e the equivalent stresses at both minimum and maximum load do not exceed the yield strength. Initial strains are not applied and the material parameters remain E and ν. The equivalent elastic-plastic stress range must be less than twice the yield stress and therefore purely elastic. Nevertheless, the fictitiously elastic calculated results do not necessarily coincide with the elastic-plastic results.

(2 - 9) $$V_e^{(n)} = \left\{\underline{x}\mid\sigma_{v,min}^{(n-1)} < f_y \;\;\wedge\;\; \sigma_{v,max}^{(n-1)} < f_y\right\}$$ Eq. 4.154

Secondly, all parts of the structure in which the equivalent stress range exceeds twice the yield stress are assigned to the plastic zone $V_{p\Delta}$.

(2 - 10) $$V_{p\Delta}^{(n)} = \left\{\underline{x}\mid\Delta\sigma_v^{(n-1)} \geq 2\,f_y\right\}$$ Eq. 4.152

To obtain results for the accumulated stresses and strains for both load conditions, the residual state for the mean load condition (midway between minimum and maximum load) is calculated. The TIV for the mean load condition can be estimated to be in the middle between fictitious elastic deviatoric stress for minimum and maximum load in the TIV space (Fig. 5, Eq. (2 - 11)). Material parameters E^* and ν^* are used in the modified elastic analysis.

(2 - 11) $$Y_{i,m} = \sigma_{i,min}^{\prime fel} + \frac{1}{2}\Delta\sigma_i^{\prime fel}$$ Eq. 4.156

This equation uses merely fictitious elastic quantities. An iterative improvement is therefore not possible.

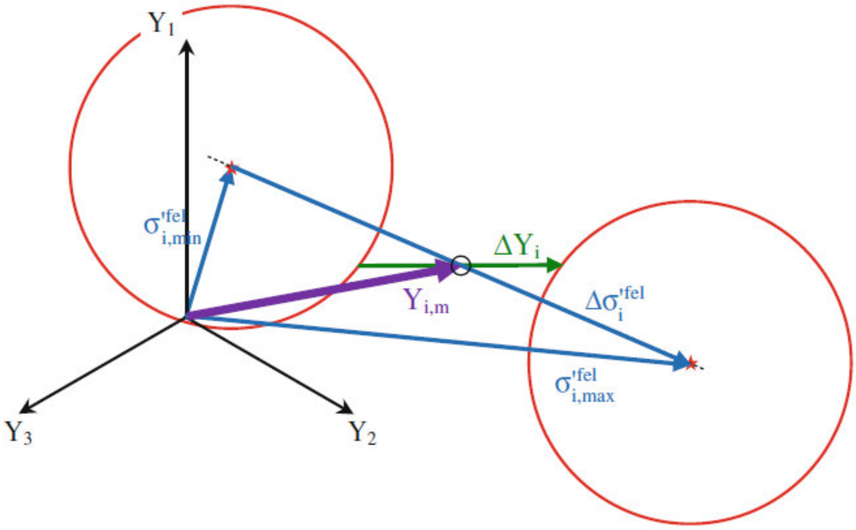


Fig. 5 Estimation of the TIV for the mean load condition at PS for a location in $V_{p\Delta}$ (Fig. 4.32 in [1])

The third zone for PS is $V_{e\Delta}$ in which the stress range is purely elastic, but the equivalent stress of either minimum, maximum or both loads exceeds the yield stress.

(2 - 12) $$V_{e\Delta}^{(n)} = \left\{\underline{x}\ |\Delta\sigma_v^{(n-1)} < 2\,f_y \wedge \left(\sigma_{v,max}^{(n-1)} > f_y \vee \sigma_{v,min}^{(n-1)} > f_y\right)\right\}$$ Eq. 4.153

If $\sigma_{v,max} > \sigma_{v,min}$ the TIV for mean load is estimated by projecting the negative deviatoric residual stress at maximum load ($-\rho_{i,max}^{\prime} = Y_{i,max}^*$) onto the yield surface of the maximum load and subtracting half of the TIV range, which is known from the previous range analysis ($\Delta Y_i = \Delta\sigma_i^{\prime fel} - \Delta\sigma_i^{\prime}$, because $\Delta\xi_i = 0$) (Fig. 6, top).

(2 - 13) $$Y_{i,m} = \sigma_{i,max}^{\prime fel} - \sigma_{i,max}^{\prime} \cdot \frac{f_y}{\sigma_{v,max}} + \frac{1}{2}\left(\Delta\sigma_i^{\prime} - \Delta\sigma_i^{\prime fel}\right)$$ Eq. 4.160

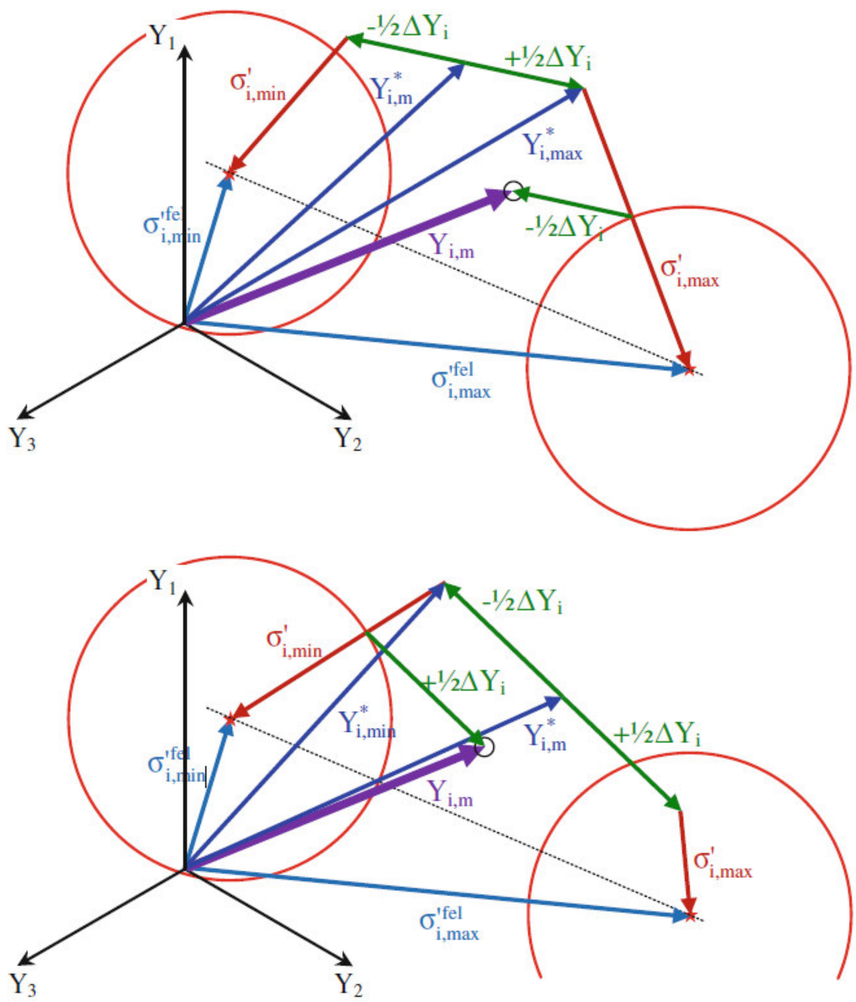


Fig. 6 Estimation of the TIV for the mean load condition at PS for a location in $V_{e\Delta}$ by projection to the maximum (*top*) and minimum (*bottom*) load condition (Fig. 4.33 and 4.34 in [1])

Otherwise (if $\sigma_{v,max} < \sigma_{v,min}$) $-\rho'_{i,min} = Y_{i,min}$ is projected onto the yield surface of the minimum load and half of the TIV range is added (Fig. 6, bottom):

(2 - 14) $$Y_{i,m} = \sigma'^{fel}_{i,min} - \sigma'_{i,min} \cdot \frac{f_y}{\sigma_{v,min}} - \frac{1}{2}\left(\Delta\sigma'_i - \Delta\sigma'^{fel}_i\right)$$ Eq. 4.162

After the TIV has been estimated, initial strains or stresses can be calculated and applied to each element of the structure in the same manner as described in the previous chapter for monotonic loading (see Eqs. (1 - 5) or (1 - 9)). The modified elastic analysis results in the residual stress state for mean load. The elastic-plastic state for mean load is obtained by superimposing on the fictitious elastic state for mean load.

(2 - 15) $$\sigma_{i,m} = \rho_{i,m} + \frac{1}{2}\left(\sigma^{fel}_{i,max} + \sigma^{fel}_{i,min}\right)$$ Eq. 4.163

(2 - 16) $$\sigma_{i,min/max} = \sigma_{i,m} \mp \frac{1}{2}\Delta\sigma_i = \sigma_{i,m} \mp \frac{1}{2}\left(\Delta\sigma^{fel}_i + \Delta\rho_i\right)$$ Eq. 4.165

When working with loadcase operations in ANSYS, a reformulation of Eq. (2 - 16) using the expression in Eq. (2 - 15) appears to be convenient.

(2 - 17)
$$\begin{aligned}\sigma_{i,max} &= \rho_{i,m} + \frac{1}{2}\left(\sigma_{i,max}^{fel} + \sigma_{i,min}^{fel}\right) + \frac{1}{2}\Delta\sigma_i \\ &= \rho_{i,m} + \frac{1}{2}\left(\sigma_{i,max}^{fel} + \sigma_{i,min}^{fel}\right) + \frac{1}{2}\left(\sigma_{i,max}^{fel} - \sigma_{i,min}^{fel}\right) + \frac{1}{2}\Delta\rho_i \\ &= \rho_{i,m} + \sigma_{i,max}^{fel} + \frac{1}{2}\Delta\rho_i\end{aligned}$$

(2 - 18)
$$\begin{aligned}\sigma_{i,min} &= \rho_{i,m} + \frac{1}{2}\left(\sigma_{i,max}^{fel} + \sigma_{i,min}^{fel}\right) - \frac{1}{2}\Delta\sigma_i \\ &= \rho_{i,m} + \frac{1}{2}\left(\sigma_{i,max}^{fel} + \sigma_{i,min}^{fel}\right) - \frac{1}{2}\left(\sigma_{i,max}^{fel} - \sigma_{i,min}^{fel}\right) - \frac{1}{2}\Delta\rho_i \\ &= \rho_{i,m} + \sigma_{i,min}^{fel} - \frac{1}{2}\Delta\rho_i\end{aligned}$$

b) Elastic Shakedown

As mentioned before, elastic shakedown occurs when the fictitious elastic equivalent stress range does not exceed twice the yield stress in any location of the structure (see Sect. 4.4 in [1]).

(2 - 19) $$\Delta\sigma_v^{fel} \leq 2f_y \quad \forall \underline{x} \in V$$ Eq. 4.56

Those parts of the structure where the equivalent stress of neither the minimum nor maximum load condition exceeds the yield stress are assigned to the elastic zone $V_e^{(n)}$. The other parts are assigned to the plastic zone $V_p^{(n)}$.

(2 - 20) $$V_e^{(n)} = \left\{\underline{x} \mid \sigma_{v,min}^{(n-1)} < f_y \;\wedge\; \sigma_{v,max}^{(n-1)} < f_y\right\}$$ Eq. 4.59

(2 - 21) $$V_p^{(n)} = \left\{\underline{x} \mid \sigma_{v,min}^{(n-1)} \geq f_y \;\vee\; \sigma_{v,max}^{(n-1)} \geq f_y\right\}$$ Eq. 4.60

Modified elastic analyses with iterative improvement can be executed to determine the accumulated state.

For elastic shakedown, in V_p the two Mises circles for minimum and maximum load form an intersection area Ω. When projecting the negative deviatoric residual stress ($-\rho_i'^{(n-1)} = Y_i^{*(n)}$) onto one of the yield surfaces, it has to be done so that the projection also lies on the edge of Ω. According to the position of the negative deviatoric residual stress, the projection then differs. The angles α_{min} and α_{max} divide the space into four regions ω_{1-4} (Fig. 7). The position of Y_i^* is described by the angles β_{min} and β_{max}.

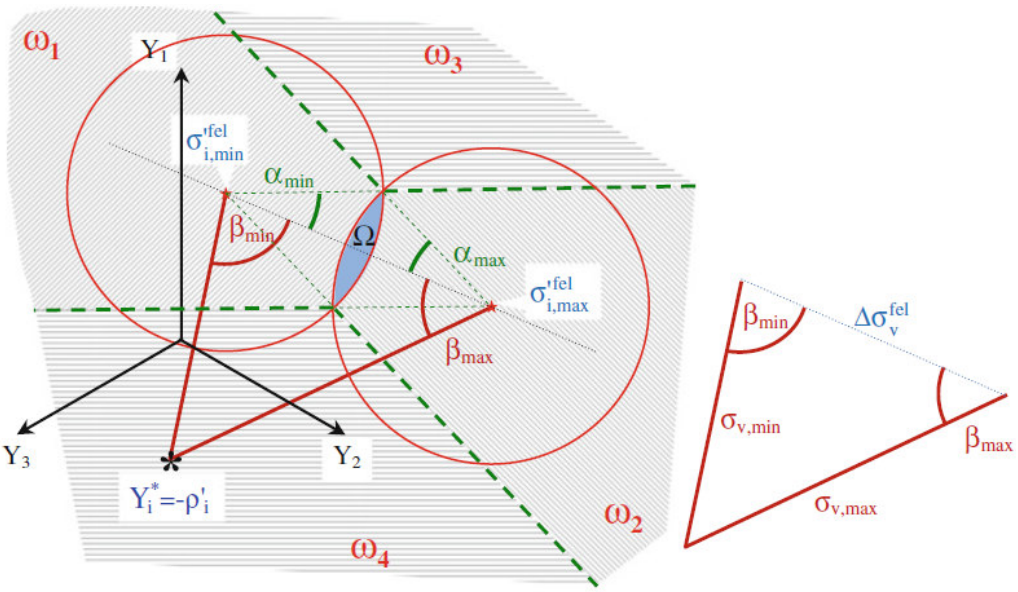


Fig. 7 Elastic shakedown; *left* intersection area Ω of the yield surfaces and position of Y* in the TIV-space; *right* isolated sketch of the triangle to determine the angles β_{min} and β_{max} (Fig. 4.12 in [1])

(2 - 22) $$\cos\alpha_{min} = \cos\alpha_{max} = \frac{\Delta\sigma_v^{fel}}{2f_y}$$ Eq. 4.63

(2 - 23) $$\cos\beta_{min} = \frac{\Delta\sigma_v^2 + \sigma_{v,min}^2 - \sigma_{v,max}^2}{2\cdot\Delta\sigma_v\cdot\sigma_{v,min}}$$ Eq. 4.64

(2 - 24) $$\cos\beta_{max} = \frac{\Delta\sigma_v^2 + \sigma_{v,max}^2 - \sigma_{v,min}^2}{2\cdot\Delta\sigma_v\cdot\sigma_{v,max}}$$ Eq. 4.65

If $\beta_{max} < \alpha_{max}$ and $\sigma_{v,max} > f_y$ then Y_i^* is in ω_1 and it is projected onto the yield surface of the maximum load.

(2 - 25) $$Y_i = \sigma_{i,max}^{\prime fel} - \sigma_{i,max}^{\prime}\left(\frac{f_y}{\sigma_{v,max}}\right)$$ Eq. 4.69

If $\beta_{min} < \alpha_{min}$ and $\sigma_{v,min} > f_y$ then Y_i^* is in ω_2 and it is projected onto the yield surface of the minimum load.

(2 - 26) $$Y_i = \sigma_{i,min}^{\prime fel} - \sigma_{i,min}^{\prime}\left(\frac{f_y}{\sigma_{v,min}}\right)$$ Eq. 4.67

If $\beta_{min} \geq \alpha_{min}$ and $\beta_{max} \geq \alpha_{max}$ then Y_i^* is in ω_4 and it is projected onto the closest intersection point of the yield surfaces. Y_i^* cannot be in ω_3.

(2 - 27) $$Y_i = \sigma_{i,min}^{\prime fel} - a\cdot\sigma_{i,min}^{\prime} + b\cdot\Delta\sigma_i^{\prime fel}$$ Eq. 4.71

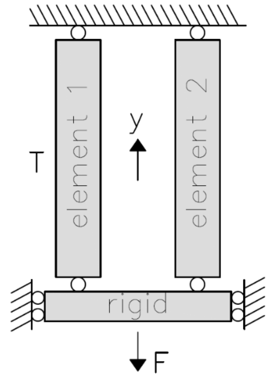
The factors a and b in (2 - 27) can be determined geometrically:

(2 - 28) $$a = \frac{f_y}{\sigma_{v,min}}\cdot\frac{\sqrt{1-\cos^2\alpha_{min}}}{\sqrt{1-\cos^2\beta_{min}}}$$ Eq. 4.73

(2 - 29) $$b = \frac{f_y}{\Delta\sigma_v}\cdot\left(\cos\alpha_{min} - \cos\beta_{min}\frac{\sqrt{1-\cos^2\alpha_{min}}}{\sqrt{1-\cos^2\beta_{min}}}\right)$$ Eq. 4.74

Although initial loads and modified material parameters are only applied in V_p, residual stresses can also appear in V_e. Superposition works in the same way as described for plastic shakedown.

2.2 Example C.1 Two-bar model



As a first example for cyclic loading, the two-bar model (first described in Sect. 2.2 in [1]) is used. The structure consists of two parallel bars with the top ends fixed and the bottom ends connected by a rigid plate (coupled degree of freedom) so the change in length must always be equal. The structure is modelled in ANSYS using two square PLANE-elements of equal size using the appropriate degree-of-freedom constraints. A constant tension (force F) is applied. The left bar is cyclically loaded and unloaded with a temperature T. The right bar prevents the left bar from free thermal expansion causing a compressive stress in the left bar and a tensile stress in the right bar, which have the same absolute value due to the equilibrium condition. Sects. 2.2 and 2.3 in [1] explain the ratcheting mechanism leading to progressive deformation.

For cyclic loading, the application of the STPZ is separated into two parts. First, it is used to obtain the elastic-plastic range values. The second part determines the accumulated values.

In [1] see also Sects. 4.3.1 for the elastic-plastic range, 4.5.1 for ES and 4.7.1 for PS.

a) Plastic shakedown C.1.a

The MEA for the elastic-plastic range values works in the same way as for monotonic loading using range values instead. In this example, the equivalent stress ranges exceed twice the yield stress in both elements. Initial stresses are applied and the modified material parameters are used. The residual range state is stored as loadcase 3 for later use. Superposition gives the elastic-plastic range state. The correct strain range is found after the first iteration.

The second part is to obtain the accumulated values for minimum and maximum load. The necessary stress values are saved as parameters. The fictitious elastic stress at mean load is calculated and saved. In this example, the TIV and the initial stress are calculated for the uniaxial stress state, as explained in Sect. 1.2. After the modified elastic analysis, the superposition is performed using Eqs. (2 - 17) and (2 - 18). The STPZ provides the exact results at both load conditions after only one MEA (see Table 1). In this case, no iterative improvement is necessary.

Table 1 Selected stress values for example C.1.a

	minimum load		maximum load	
	element 1	element 2	element 1	element 2
	$\sigma_{y,min}$	$\sigma_{y,min}$	$\sigma_{y,max}$	$\sigma_{y,max}$
1. MEA	145.00	-45.00	-75.00	175.00

b) Elastic shakedown C.1.b

The temperature T is lowered so that the equivalent stress range is less than twice the yield stress. Therefore, the elastic-plastic stress range is known from the fictitious elastic analysis. For elastic shakedown the angles α, β_{min} and β_{max} are needed to tell in which region ω_i the negative deviatoric residual stress is positioned. Accordingly, the TIV is estimated. In the first MEA $-\rho'$ is in ω_1 for both elements. In the second MEA $-\rho'$ changes to ω_2. The results after the second MEA are exact. An incremental analysis calculating 30 cycles with 20 substeps per half cycle produces almost the exact result (see Table 2 and Fig. 8).

Table 2 Selected stress values for example C.1.b

	minimum load		maximum load	
	element 1	element 2	element 1	element 2
	$\sigma_{y,min}$	$\sigma_{y,min}$	$\sigma_{y,max}$	$\sigma_{y,max}$
1. MEA	135.50	-35.50	-54.50	154.50
2. MEA	140.25	-40.25	-49.75	149.75
Incr. analysis	140.15	-40.15	-49.61	149.61

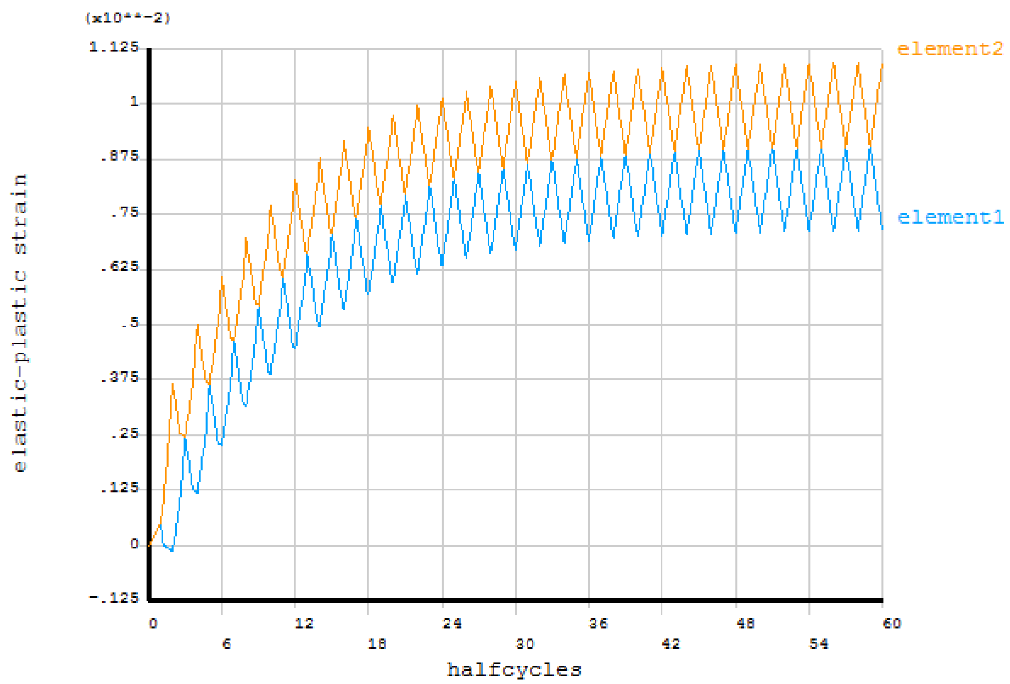
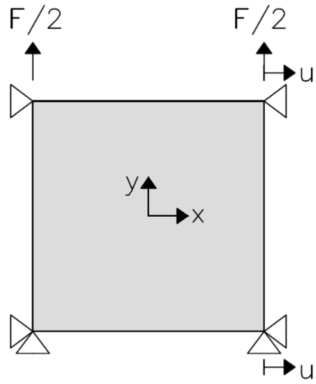


Fig. 8 Development of strains in incremental analysis for elastic shakedown for example C.1.b

2.3 Example C.2 Multiaxial ratcheting



For this example, the ratcheting mechanism is explained in Sect. 2.5.2 in [1]. The structure is similar to example M.2 with the following adjustment: the degrees of freedom in the y-direction for the top two nodes are not inhibited. Instead, a force $F/2$ in the y-direction is applied at both nodes, causing the so-called primary stress σ_p. The displacement-controlled load is applied cyclically, alternating between u_{min} and u_{max} causing the secondary stress range $\Delta\sigma_t$.

In [1] see also Sects. 4.3.2 for the elastic-plastic range, 4.5.2 for ES and 4.7.2 for PS.

a) Plastic shakedown C.2.a

The elastic limit for the strain range and therefore the condition for plastic shakedown (see Eq. (2 - 2)) are calculated merely from the material parameters, because unit lengths are used.

(2 - 30) $$\varepsilon^{el} < \frac{2f_y}{E} = \frac{2 \cdot 100}{100\,000} = 0.002$$ Eq. 4.153

The elastic limit is exceeded when choosing load parameters $u_{min} = 0$ and $u_{max} = 0.0025$. The algorithm of the STPZ is the same as in example C.1 extended for a second component in the x-direction. The correct solution is obtained after one iteration each (range and mean load), adding up to the computational effort of four linear analyses.

Fig. 9 shows the development of the strain components in an incremental analysis. More than one hundred cycles were calculated to reach the state of shakedown. The computational effort in terms of the numbers of analyses is significantly greater than for using the STPZ.

Table 3 Selected stress and strain values for example C.2.a

	$\sigma_{x,min}$	$\sigma_{x,max}$	$\varepsilon_{y,min}$	$\varepsilon_{y,max}$
1. MEA	-71.25	131.25	7.27E-03	7.03E-03

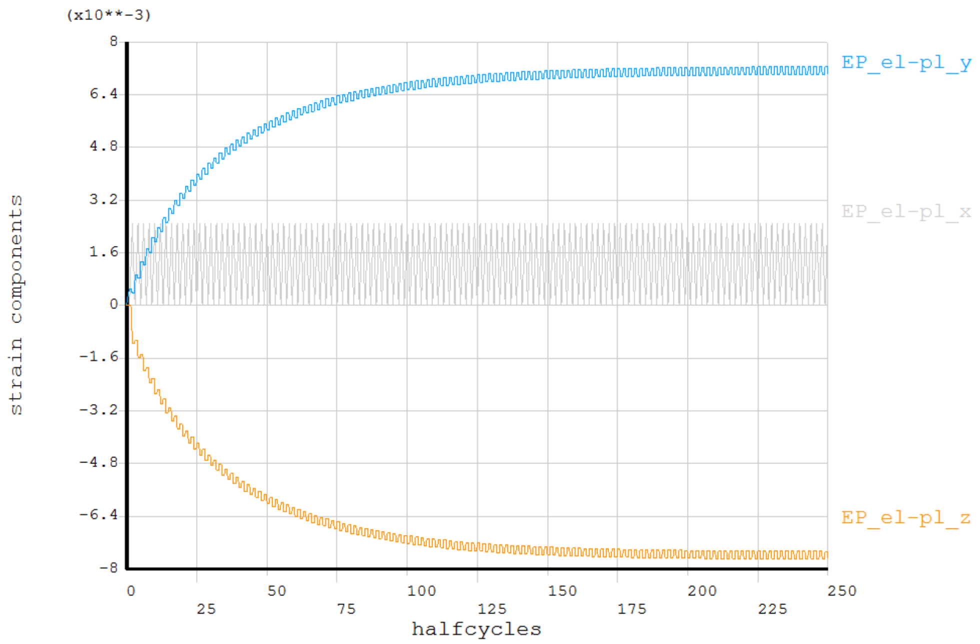


Fig. 9 Development of strain components in incremental analysis for plastic shakedown in example C.2.c

b) Elastic shakedown C.2.b and C.2.c

The first set of loading parameters $u_{min} = 0.0010$ and $u_{max} = 0.0025$ is chosen. This leaves the strain range to be purely elastic ($\Delta\sigma_v < 2f_y$) but the equivalent stress at maximum load exceeds the yield strength. The element is therefore assigned to zone V_p. The stress and strain ranges can be taken directly from the fictitious elastic analysis. The STPZ is used to obtain the minimum and maximum load state. The values for the angles α, β_{min} and β_{max} (Eqs. (2 - 22), (2 - 23) and (2 - 24)) are calculated in order to locate the negative deviatoric residual stress in one of the regions ω_1, ω_2 or ω_4. Accordingly the TIV is estimated following one of the Eqs. (2 - 25), (2 - 26) or (2 - 27). In this case $-\rho'$ is in ω_1. The application of the initial stress and the superposition work as before. Table 4 lists the development of relevant stress and strain values after each of the first six MEAs. After the third MEA, the values obtained from an incremental analysis are almost met. After the sixth MEA, the results are the same.

Table 4 Selected stress and strain values for example C.2.b

	$\sigma_{x,min}$	$\sigma_{x,max}$	$\varepsilon_{y,min}$	$\varepsilon_{y,max}$
1. MEA	-20.46	129.54	3.91E-03	3.91E-03
2. MEA	-25.98	124.02	6.98E-04	6.98E-04
3. MEA	-26.71	123.29	3.99E-04	3.99E-04
4. MEA	-26.81	123.19	3.57E-04	3.57E-04
5. MEA	-26.83	123.17	3.51E-04	3.51E-04
6. MEA	-26.83	123.17	3.50E-04	3.50E-04
Incr. analysis	-26.83	123.17	3.50E-04	3.50E-04

In addition, a second set of loading parameters is examined in C.2.c: $u_{min} = 0.0002$ and $u_{max} = 0.0021$. Again, elastic shakedown occurs and the Mises equivalent stress of one of the extreme loads exceeds the yield strength. In the first MEA $-\rho'$ is located in ω_1 but in the second MEA it is located in ω_4, meaning it is projected onto the vertex of Ω. A further improvement is not possible because the position of the vertex does not change.

Table 5 Selected stress and strain values for example C.2.c

	$\sigma_{x,min}$	$\sigma_{x,max}$	$\varepsilon_{y,min}$	$\varepsilon_{y,max}$
1. MEA	-63.25	126.75	3.46E-03	3.46E-03
2. MEA	-65.50	124.50	2.06E-03	2.06E-03
Incr. analysis	-65.49	124.49	2.05E-03	2.05E-03

The incremental analysis of this problem shows that more than 250 cycles have to be calculated to reach the state of shakedown (Fig. 10).

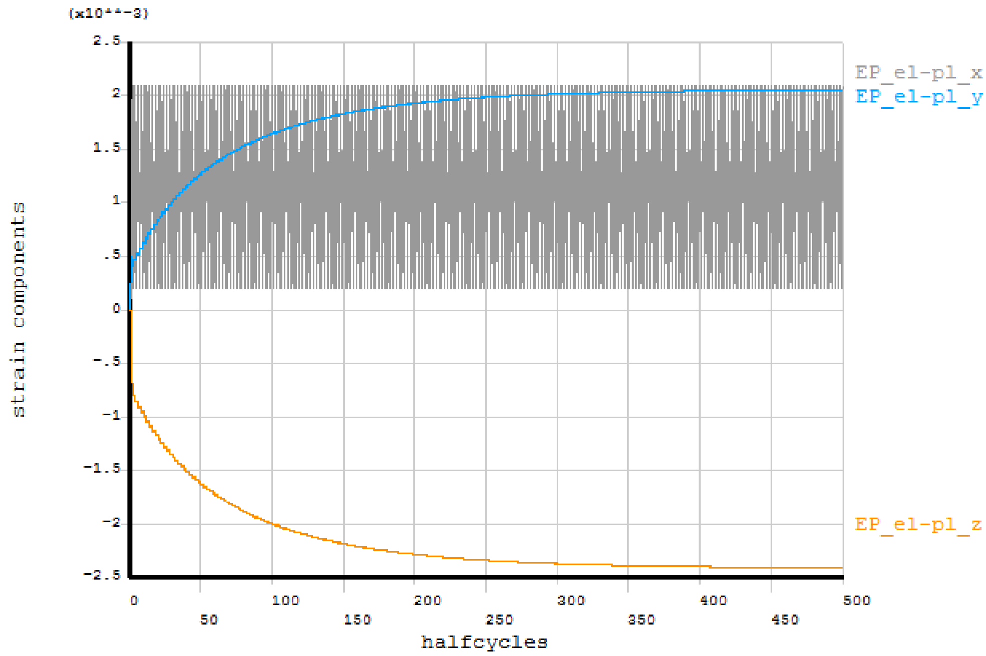
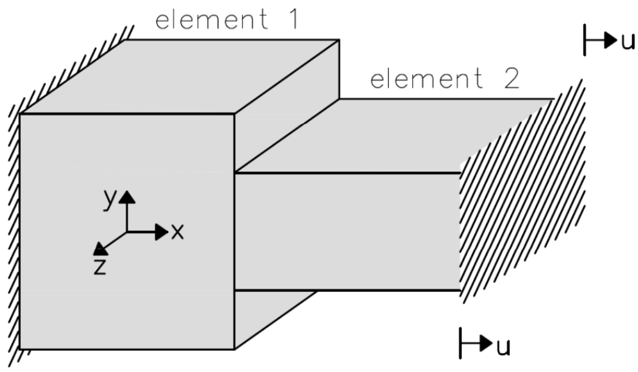


Fig. 10 Development of strain components in incremental analysis for elastic shakedown in example C.2.c

2.4 Example C.3 Tension bar



In this last example, the model already introduced in example M.3 is used. Although no ratcheting mechanism appears for cyclic loading as no additional primary stress is applied, the model is chosen because again the algorithm is extended to the three normal stress components and, in addition, it can be observed how the TIV is estimated for elements assigned to zone $V_{e\Delta}$ and how the initial stresses and modified material law are reversed for elements in V_e.

a) Plastic shakedown C.3.a

For plastic shakedown the load parameters $u_{min} = 0.001$ and $u_{max} = 0.0065$ are chosen. The elastic-plastic range state is correctly obtained after one MEA. Although the equivalent stress range does not exceed twice the yield stress for element 1, it does for element 2. The maximum equivalent stress is saved for later comparison.

Table 6 Mises-equivalent stress at extreme loads before the first four MEAs for example C.3.a

	element 1		element 2	
	$\sigma_{v,min}$	$\sigma_{v,max}$	$\sigma_{v,min}$	$\sigma_{v,max}$
f. el.	37.46	243.49	63.63	413.61
1. MEA	27.19	117.58	39.20	180.04
2. MEA	46.58	98.24	74.95	144.51
3. MEA	47.66	97.17	77.01	142.50

In both elements, the fictitiously elastic calculated equivalent stresses at maximum load exceed the yield stress $f_y = 100$ (see Table 6). Therefore element 1 is assigned to $V_{e\Delta}$ and element 2 to $V_{p\Delta}$ in the first MEA. The TIV is estimated accordingly, initial stresses are applied and the material parameters are changed. For the second MEA the assignment remains the same, but after the second iteration, the equivalent stress at maximum load in element 1 is less than the yield stress. For the third MEA, element 1 is in V_e. The initial stresses have to be deleted and the material parameters are reversed to E and ν. The result of the third MEA cannot be improved any further (see Table 7). The quality of the results for stress values compared to an incremental analysis differ according to the direction and the load state. For the incremental analysis 50 substeps per half cycle were calculated, but the state of shakedown is obtained after one cycle.

Table 7 Selected stress values for example C.3.a

	minimum load				maximum load			
	element 1		element 2		element 1		element 2	
	$\sigma_{x,min}$	$\sigma_{z,min}$	$\sigma_{x,min}$	$\sigma_{z,min}$	$\sigma_{x,max}$	$\sigma_{z,max}$	$\sigma_{x,max}$	$\sigma_{z,max}$
1. MEA	-22.01	7.04	-44.03	-14.08	100.84	-27.97	201.69	55.94
2. MEA	-40.92	8.42	-81.84	-16.84	81.94	-26.59	163.87	53.18
3. MEA	-42.18	8.66	-84.36	-17.32	80.68	-26.35	161.35	52.70
4. MEA	-42.18	8.66	-84.36	-17.32	80.68	-26.35	161.35	52.70
Incr. analysis	-48.08	13.44	-96.17	-26.87	80.66	-28.06	161.32	56.13

b) Elastic shakedown C.3.b

For elastic shakedown the load parameters examined are $u_{min} = 0.001$ and $u_{max} = 0.004$. The equivalent stress range is less than twice the yield stress in both elements.

The fictitious elastic equivalent stress at maximum load exceeds the yield stress in both elements. They are assigned to zone V_p in the first MEA. The negative deviatoric residual stress is in ω_1. In the second MEA, the equivalent stress is less than the yield stress at both extreme loads in element 1. It is now assigned to V_e. The initial stresses from the previous MEA are deleted and material parameters are reversed. The result can be improved further in later MEAs but element 2 remains in V_e. An incremental analysis produces the exact result after one cycle.

Table 8 Selected stress values for example C.3.b

	minimum load				maximum load			
	element 1		element 2		element 1		element 2	
	$\sigma_{x,min}$	$\sigma_{z,min}$	$\sigma_{x,min}$	$\sigma_{z,min}$	$\sigma_{x,max}$	$\sigma_{z,max}$	$\sigma_{x,max}$	$\sigma_{z,max}$
1. MEA	-16.26	-4.96	-32.52	9.92	86.59	-22.07	173.18	44.13
2. MEA	-35.30	-1.96	-70.60	3.93	67.54	-19.07	135.09	38.14
3. MEA	-35.00	-3.36	-70.00	6.72	67.85	-20.47	135.70	40.93
8. MEA	-34.51	-6.17	-69.02	12.34	68.34	-23.27	136.68	46.55
Incr. analysis	-34.42	-6.82	-68.84	13.65	68.43	-23.93	136.86	47.86

3 Advantages and disadvantages of the STPZ

The advantage of the simplified theory of plastic zones has clearly been shown. The analysis of cyclically loaded structures often demands high computational effort. Numerous cycles with many substeps and equilibrium iterations have to be calculated to obtain stress and strain values at the state of shakedown. The STPZ delivers a good approximation after only a few linear analyses.

Usually one does not know the amount of iterations needed to approximate the exact result. Therefore the analyst has to examine changes in the results after each step and stop when the differences become insignificant. The quality of the result can further be affected by directional stress redistribution. The quality of results for individual stress components can vary within one analysis. These disadvantages have to be considered.

It is recommended to additionally study [1], where more complex structures are analysed with the STPZ and the results are discussed. The examples also include temperature-dependent material and multilinear hardening.

4 References

[1]. Hübel, Hartwig: Simplified Theory of Plastic Zones - Based on Zarka's Method. Springer International Publishing, Cham, Switzerland (2017)

[2]. Hübel, Hartwig: Vereinfachte Fließzonentheorie - Auf Grundlage der Zarka Methode. Springer Vieweg, Wiesbaden (2015)

[3]. ANSYS Release 14.5, ANSYS Inc. Canonsburg, USA (2012)

List of Annexes

Examples for monotonic loading

- M.1 Element with uniaxial stress state, STPZ
- M.1 Element with uniaxial stress state, incremental analysis
- M.2 Element with plane stress state, STPZ
- M.2 Element with plane stress state, incremental analysis
- M.3 Tension bar, STPZ
- M.3 Tension bar, incremental analysis

Examples for cyclic loading

- C.1.a Two-bar model, PS, STPZ
- C.1.a Two-bar model, incremental
- C.1.b Two-bar model, ES, STPZ
- C.1.b Two-bar model, incremental
- C.2.a Multiaxial ratcheting, PS, STPZ
- C.2.a Multiaxial ratcheting, incremental
- C.2.b Multiaxial ratcheting, ES-ω_1, STPZ
- C.2.b Multiaxial ratcheting, incremental
- C.2.c Multiaxial ratcheting, ES-ω_4, STPZ
- C.3.a Tension bar, PS, STPZ
- C.3.a Tension bar, incremental
- C.3.b Tension bar, ES, STPZ
- C.3.b Tension bar, incremental

All annexes are given as individual .txt-files.