Influence of the central force field on the flow dynamics for non-rotating or slow rotating spherical shell

Onset of the convection

Philippe Beltrame

[1] Universität Augsburg, Germany

Pascal Chossat

CIRM, Marseille, France

Outline

Bifurcation (non-rotating case)

- Codimension 1 bifurcation
- Mode interaction

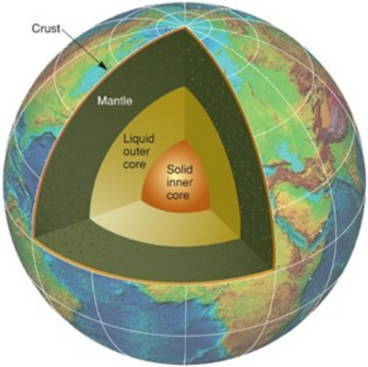
Heteroclinic cycles

- self-adjoint
- GEOFLOW degeneracy

Slow rotation

- Self-adjoint
- GEOFLOW degeneracy

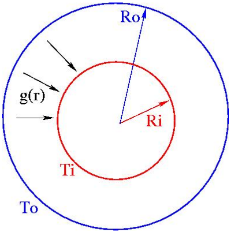
Geophysical Framework



- Large scale motion in stars and planets due to:
  - Convection (hot core)
  - Rotation
- Main characteristics
  - Spherical shell
  - Central force field
    - $1/r^2$: Earth's outer core
    - r: Earth's mantle
- Questions ?
  - Motion of Earth's Plates (Earth's mantle)
  - Motion of Magnetic pole (Earth's outer core)

Modelling & approach

- Viscous, homogeneous incompressible fluid,
- Central gravity force $g(r)\mathbf{u_r}$,
- Uniformly heated inner sphere, Uniformly cooled outer sphere.



- no rotation
- thin axis neglected

$\longrightarrow$ $O(3)$-invariant problem

1. Onset of convection of the basic flow ('pure' thermal diffusion)
2. small rate rotation: perturbation of convection flow (forced symmetry breaking)

$O(3)$-Equivariant bifurcation theory

- In 80's many studies devoted to dynamical behavior related to spherical symmetry (Busse, Chossat, Golubitsky, Lauterbach...)
- Generically, near the convection onset are expected only stationary or travelling waves solutions

  Gaeta and Rossi, J. Math. Phys, 25(6), 1671-1673, 1984

- For the $(\ell, \ell+1)$ interaction of the spherical modes, robust heteroclinic cycles can occur because of symmetry. These cycles are reminescent of the quasi-periodic magnetic pole reversal.

  Guckenheimer and Holmes, Math. Proc. Camb. Phil. Soc., 103: 189-192, 1988

Robust heteroclinic cycles

- The $(1, 2)$ interaction was thoroughly numerically and theoretically studied for a $1/r^2$ force field.

  Friedrich and Haken, Am. Phys. Soc., 34(3): 2100-2120, 1986

  Armbruster and Chossat, Physica D, 50: 155-176, 1991

- The $(2, 3)$ interaction was studied for a $1/r^2$ and $1/r^5$ force field. For this last field, the heteroclinic cycles appear only near a small critical Prandtl number (not possible with GEOFLOW.)

  Chossat and Guyard, J. Nonlinear Sci., 6:201-238, 1996

  Beltrame and Egbers, Prog. in Turbulence, Peinke, Kittel (Eds): 133-136, 2005

$\rightarrow$ Onset of the $(3, 4)$ mode interaction
Comparison betwenn GEOFLOW-experiment and geophysical framework: 2 different force fields: $1/r^2$ and $1/r^5$.

Outline

Bifurcation (non-rotating case)

- Codimension 1 bifurcation
- Mode interaction

Heteroclinic cycles

- self-adjoint
- GEOFLOW degeneracy

Slow rotation

- Self-adjoint
- GEOFLOW degeneracy

Governing equations for the onset of convection

Boussinesq's approximation

$$\frac{\partial \mathbf{v}}{\partial t} = Pr\left(-\nabla p + \Delta \mathbf{v} - Ra g^*(r)\Theta \mathbf{u_r}\right) - \mathbf{v}.\nabla \mathbf{v} \qquad (1)$$

$$\frac{\partial \Theta}{\partial t} = \left(\Delta \Theta + v_r T^*(r)\right) - \mathbf{v}.\nabla \Theta \qquad (2)$$

$$\nabla.\mathbf{v} = 0 \qquad (3)$$

Homogeneous Boundary condition

- $g^*(r) = r^n$, $n = -5, -2, 1$.
- $T^*(r) = r^{-2}$

$\rightarrow$ for $n = -2$: linear part is a self-adjoint operator

Relevant dimensionless numbers:

- Ra: Rayleigh number (buoyancy force $(T_2 - T_1)$)
- Pr: Prandtl number (viscosity/thermal diffusivity)
- $\eta = R_i/R_o$: aspect ratio

Spherical Harmonics decomposition

$\mathbf{z} = (u, \Theta)) \in H$, H an Hilbert space.

$$\frac{d\mathbf{z}}{dt} = F(\eta, Ra, \mathbf{z}) \tag{4}$$

$$= DF_{(0,0)}(\eta, Ra)\mathbf{z} - M(\mathbf{z}, \mathbf{z}) \tag{5}$$

$O(3)$-equivariance

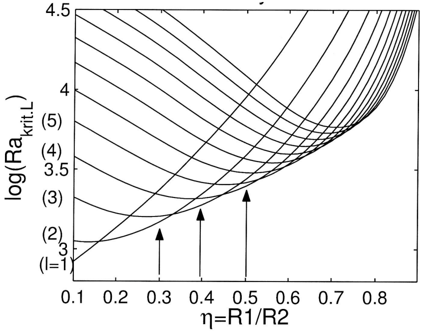
$$T_g F(\eta, Ra, \mathbf{z}) = F(\eta, Ra, T_g\mathbf{z}), \qquad g \in SO(3)$$

$$O(3) = SO(3) \oplus Z_2^c$$

- H: sum of representation of irreductible representation of degree ℓ spanned by spherical functions $T_m^\ell(\theta, \phi)$ with $(-\ell \leq m \leq \ell)$
- $z = \sum_\ell \sum_{m=-\ell}^{\ell} z_m^\ell(r) T_m^\ell(\theta, \phi)$
- Z_2^c natural Antipodal action: $S \cdot T_m^\ell = (-1)^\ell T_m^\ell$

Linear Stability: results

- Eigenspace $\simeq$ irreductible representation of $O(3)$ of degree ℓ
- Center Manifold reduction on the Eigenspace ($2\ell + 1$ dimension)



- Numerical Method
  - Pseudo-spectral
  - QZ method (eigenvalue)
- Dielectrophoretic force
  - Stat. bif.
  - ℓ increases with η

Similar results for all force fields:
(3,4) mode interaction for the aspect ratio $\eta \simeq 0.45$

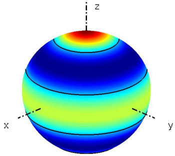
Generic bifurcation

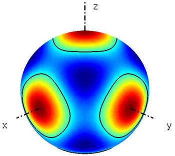
Let Σ be an isotropy subgroup for the action of $O(3)$ in V^ℓ, then the invariant space

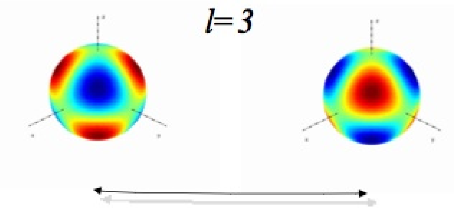
$$Fix(\Sigma) = \{x \in V^\ell / x \text{ is fixed by } \Sigma\}$$

- Even mode: transcritical bifurcation
  - $\partial_\mathbf{t}\mathbf{x_4} = \mu_2\mathbf{x_4} + c_{44}P_{44}(\mathbf{x_4}, \mathbf{x_4}) + h.o.t.$
  - $\ell = 2$ one unstable transcritical branch axisymmetric
  - $\ell = 4$ two unstable transcritical branches: axisymmetric and cubic
- Odd mode: pitchfork bifurcation
  - Odd mode: $\partial_\mathbf{t}\mathbf{x_3} = \mu_1\mathbf{x_3} + c_{333}P_{333}(\mathbf{x_3}, \mathbf{x_3}, \mathbf{x_3}) + h.o.t.$ (Antipodal symmetry)
  - $\ell = 3$: 3 supercritical branches: axisymmetric, 3-fold dihedral symmetry, tetrahedral symmetry
    Only one branch is stable.

Examples





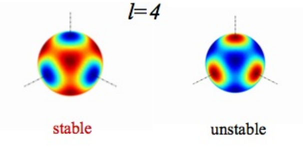


Same solution (tetrahedral symmetry)

Self-adjoint degeneracy

$c_{44} \simeq 0$ (quadratic term of even mode).
Consequences:

- two sided branches
- One branch is stable
  - $\ell = 2$ the axisymmetric one
  - $\ell = 4$ the cubic one



- Secondary bifurcations (codimension 2 bifurcation)

C. Geiger, G. Dangelmayr et al. Fields Institute Com. Vol. 5, AMS, 1993

Self-adjoint degneracy for $g* \neq r^{-2}$?
$\rightarrow$ coefficient computation vs *Pr*

Mode interaction

Amplitude equation

$$\partial_t \mathbf{x_3} = c_3 \mathbf{x_3} + c_{34} P_{34}(\mathbf{x_3}, \mathbf{x_4}) + h.o.t. \quad (6)$$
$$\partial_t \mathbf{x_4} = c_4 \mathbf{x_4} + c_{44} P_{44}(\mathbf{x_4}, \mathbf{x_4}) + c_{33} P_{33}(\mathbf{x_3}, \mathbf{x_3}) + h.o.t. \quad (7)$$

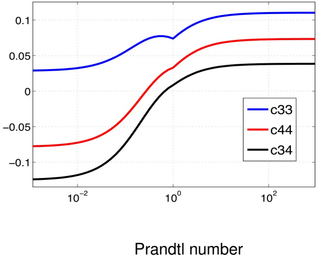
- self adjoint: $\rightarrow c_{44} = 0$ independently of Pr

  Chossat and Guyard, J. Nonlinear Sci., 6:201-238, 1996

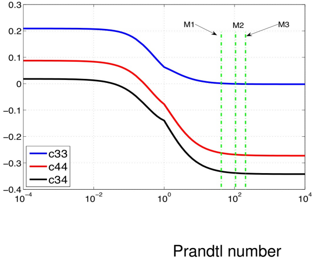
  $\rightarrow$ pitchfork bifurcation of even mode with two opposite sol.
- GEOFLOW: generically $c_{xy} \neq 0$

Coefficient Degeneracy

coeff. var. for $g^* = r$



coeff. var. for $g^* = 1/r^5$



Beltrame and Egbers, Proc. Appl. Math. Mech. 4: 474-475, 2004

GEOFLOW ($g* = 1/r^5$):

- Self-adjoint $c_{44} \simeq 0$ only in the neighborhood of $Pr_c < 1$
- GEOFLOW degeneracy for $Pr > 35$: $c_{33} \simeq 0$
  $\rightarrow$ "pure" mode 3 sol. can exist without mode 4 (antisym.)

Outline

Bifurcation (non-rotating case)

- Codimension 1 bifurcation
- Mode interaction

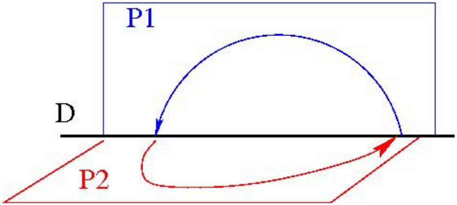
Heteroclinic cycles

- self-adjoint
- GEOFLOW degeneracy

Slow rotation

- Self-adjoint
- GEOFLOW degeneracy

Simple heteroclinic cycle



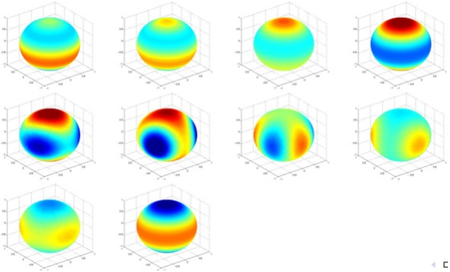
Theorem : for nearly self-adjoint ($T* \simeq g*$) the existence of heteroclinic cycles connecting opposite steady-state of even mode where

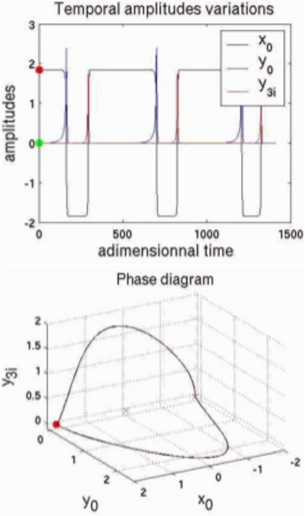
- D: axisymmetric of even modes
- P1: isotropy of O(2)
- P2: dihedral type D_n

Chossat and Guyard, J. Nonlinear Sci., 6:201-238, 1996

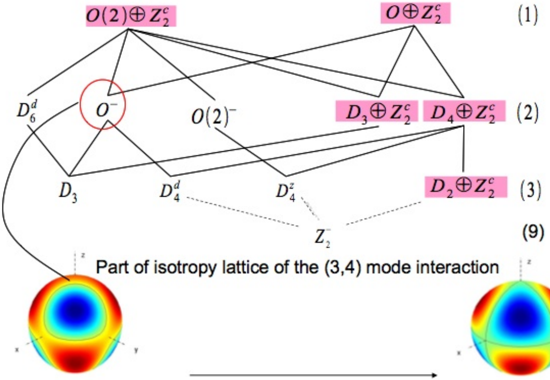
Self-adjoint: Simple cycle: the (2,3) Mode interaction

- Exsitence: direct application of last theorem
- $P2 = Fix(D_6^d)$



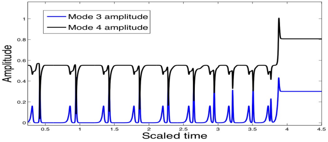


Self-adjoint: Lattice of isotropy types: (3,4) mode interaction

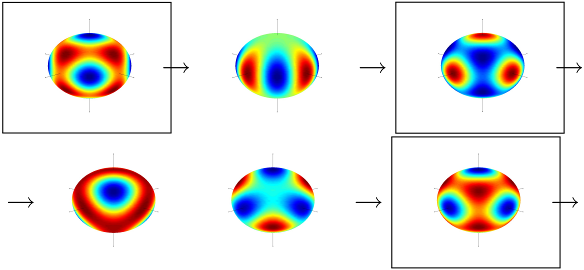


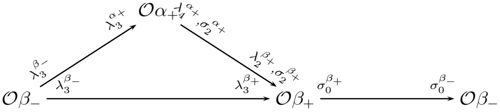
Part of isotropy lattice of the (3,4) mode interaction

The $(3, 4)$ interaction is an exception of the self-adjoint theorem.



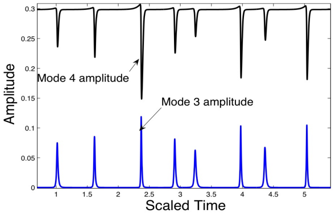
"Simple" cycle between opposite cubic solutions involving the plane $Fix(D_4 \oplus Z_2^c)$ and $(Fix(O^-))$.





Chossat and Beltrame, in progress

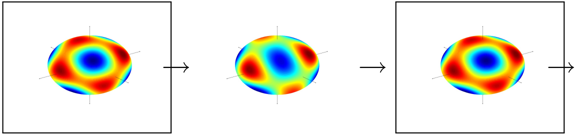
self-adjoint: Homoclinic cycle



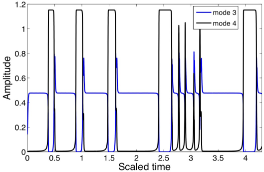
Connections between one cubic symmetric sol. During the transition the $\pi/2$ rotation symmetry and planar reflection are broken:

$O \longrightarrow D_2^z \longrightarrow O$

$dim(Fix(D_2^z)) = 5$

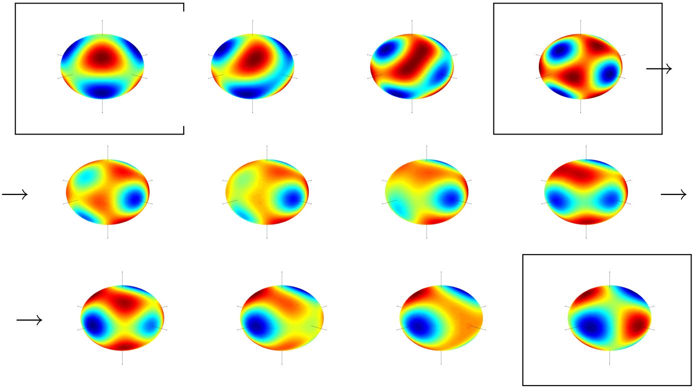


GEOFLOW: Heteroclinic cycles

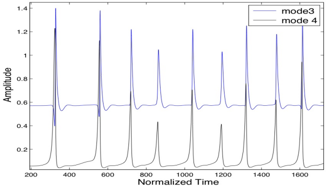


Connections between cubic and tetrahedral solutions.

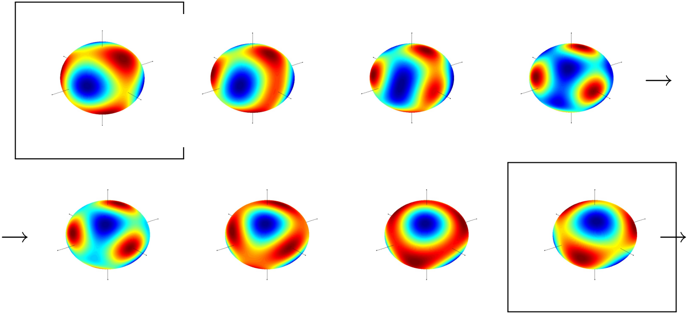
1. transition in the 3D space: $Fix(D_4^d)$
2. almost of symmetries are broken(remains (Z_2^c)) during the back connection.



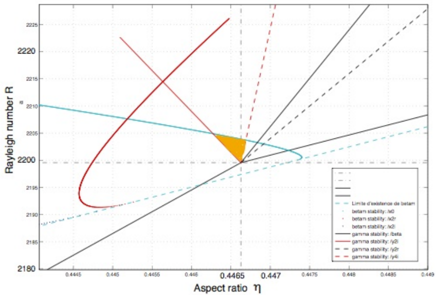
GEOFLOW: Homoclinic cycle

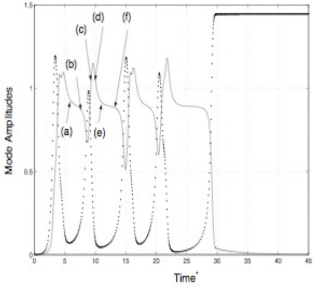


Connections of 'near' tetrahedral solutions (D_4^d). Presence of a drift: the new axis of the sol. is random. In the direct simulation this cycle is terminated by the cubic sol.



GEOFLOW: Existence domain of heteroclinic cycle



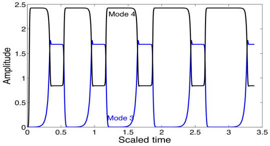


Domain of existence of het. cycle

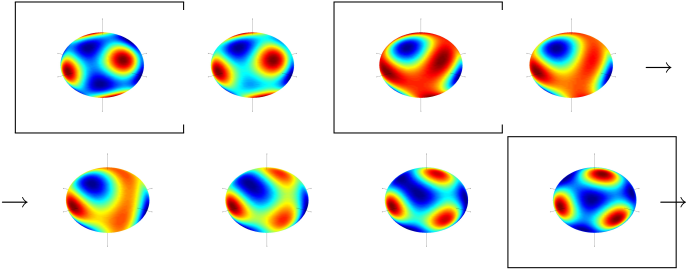
Time evolution for the direct simulation

Problem: cubic states are "away" from turning point of bifurcated branch.

GEOFLOW: Heteroclinic cycle submaximal



Connections between cubic and the "mixed" tetraheral solution.
1. transition in $Fix(D_2)$ 2. transition in $Fix(Z_2^-)$?



Outline

Bifurcation (non-rotating case)
- Codimension 1 bifurcation
- Mode interaction

Heteroclinic cycles
- self-adjoint
- GEOFLOW degeneracy

Slow rotation
- Self-adjoint
- GEOFLOW degeneracy

Small rate rotation

$$\frac{\partial \mathbf{v}}{\partial t} = Pr\left(-\nabla p + \Delta \mathbf{v} - Rag^*(r)\Theta \mathbf{u_r} + \mathbf{Tav} \times \mathbf{e_z}\right) - \mathbf{v}.\nabla \mathbf{v} \quad (8)$$

$$\ldots \quad (9)$$

Assumption: rotation rate small enough means the center manifold is still relevant
$\rightarrow$ addition of small terms in the linear part of amplitude equation

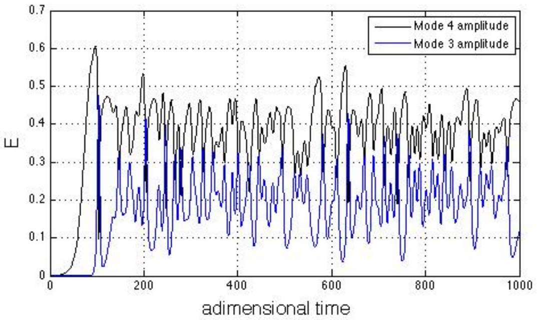
$$\partial_\mathbf{t} \mathbf{x_k} = (\mu_i + j\epsilon\omega k + O(\epsilon^2))\mathbf{x_k} + \ldots \quad (10)$$

We have an negative drift ($\omega < 0$) for both cases: self-adjoint and Geoflow according to numerical results.

Travnikov, Beltrame et al., AIP, 733, 45-57 (2004).

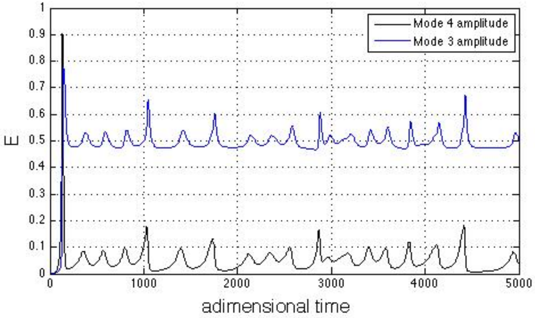
Self-adjoint cycle

Hetroclinic cycles –> Generalized HC connecting Rotating Waves instead Steady-States.



Geoflow cycle

- remains only the homo. cycle involving tetrahedral state
- amplitude of burst smaller
- mode 4 amplitude does not vanish



Concluding Remarks

- r, r^{-5}: degeneracy occurs only near critical Prandtl number
- $\frac{1}{r^2}$: degeneracy for all Prandtl number

- self-adjoint degeneracy: complex dynamics involving SS of even modes
  - generically axisymmetric solution
  - $(3,4)$ mode interaction: axisymmetric and cubic solution
- GEOFLOW degeneracy: complex dynamics involving the cubic SS and the tetrahedral SS.

Rotation:

- self-adjoint degeneracy: complex dynamics involving RW of even modes
- GEOFLOW degeneracy: complex dynamics involving RW of tetrahedral symmetry